Please show work or give reasoning for every answer. I need some evidence that you understand the topics. (No credit will be given for correct answers without an indication of how you arrived at your conclusion.)

Please show how to use derivatives to answer each question.

If you use a memorized or programmed formula, please write down the formula that you are using.

1. Suppose we are given the following data for the functions \( f \) and \( g \) and their derivatives.

\[
\begin{align*}
  f(2) &= 4 & g(2) &= 1 \\
  f(3) &= 2 & g(3) &= 2 \\
  f(4) &= 3 & g(4) &= 5 \\
  f'(2) &= -2 & g'(2) &= 2 \\
  f'(3) &= 1 & g'(3) &= 4 \\
  f'(4) &= 0 & g'(4) &= 0
\end{align*}
\]

Compute the following, or state what additional information is needed to compute the answer.

(a) If \( H(x) = f(x) + 5g(x) \), find \( H'(3) \).

(b) If \( R(x) = f(x)g(x) \), find \( R'(3) \).

(c) If \( S(x) = \frac{f(x)}{g(x)} \), find \( S'(3) \).

(d) If \( T(x) = f(g(x)) \), find \( T'(3) \).

(e) If \( h(x) = \ln(g(x)) \), find \( h'(3) \).
2. In lab, you investigated the family of functions described by
\[ f(x) = x^n e^{-bx}, \]
where \( n \) and \( b \) are positive constants. Find the \( x \) value(s) of any critical points. You may use the Mathematica output shown.

\[
\begin{align*}
\text{In[1]} &= \text{Clear}[f, x, b, n] \\
f[x_] &= x^n e^{-bx} \\
der &= D[f[x], x] \\
\text{Factor}[der] &
\end{align*}
\]

\[
\begin{align*}
\text{Out[2]} &= e^{-bx} x^n \\
\text{Out[3]} &= e^{-bx} n x^{-1+n} - b e^{-bx} x^n \\
\text{Out[4]} &= -e^{-bx} x^{-1+n} (-n + b x)
\end{align*}
\]

3. What function will be the output when the following Mathematica commands are run?

\[
\begin{align*}
\text{Clear}[f, x, h] \\
f[x_] &= x^3; \\
\text{Limit}[ (f(x+h)-f(x))/h, h \to 0 ]
\end{align*}
\]

4. Consider the graph of the equation \( y^3 + xy = 10 \). Use implicit differentiation to find a formula the slope of the line tangent to this curve at a point \((x, y)\).

5. Suppose sand is falling into a conical pile in such a way that the volume of the pile (in \( m^3 \)), \( V \), at any time is related to the height of the pile (in \( m \)), \( h \), according to
\[ V = \frac{1}{3}\pi h^3. \]

If the height of the pile is increasing at a rate of \( \frac{dh}{dt} = 2 \), at what rate is the volume increasing when the pile is one meter high?
For these questions, show how to use derivatives to obtain the exact answer.

6. (a) Find \( f'(x) \) and \( f''(x) \) for \( f(x) = x^5 - \frac{1}{2}x^2 \).

(b) Identify the \( x \)-value of any critical point(s) of \( f \).

(c) Where is the function \( f(x) = x^5 - \frac{1}{2}x^2 \) increasing?

(d) Where is the function \( f(x) = x^5 - \frac{1}{2}x^2 \) concave up?

(e) Where does the function \( f(x) = x^5 - \frac{1}{2}x^2 \) have a local maximum?

(f) Where does the function \( f(x) = x^5 - \frac{1}{2}x^2 \) have an inflection point?

(g) Find the absolute (global) maximum of \( f(x) = x^5 - \frac{1}{2}x^2 \) on the interval \(-10 \leq x \leq 10\).
7. Suppose you want to enclose a 1000 square foot rectangular region, using shrubs along one side and fencing along the other three sides. Shrubs cost $30 per foot and fencing costs $10 per foot. Your goal is to find the dimensions which result in the lowest total cost for the project (the cost of fencing plus shrubs).

(a) Sketch a diagram, using variables to label all the lengths involved.

(b) What quantity do you want to minimize/maximize? Write a formula for this quantity (in terms of the variables in your diagram).

(c) Write a formula for this quantity as a function of a single variable.

(d) Find the dimensions of the region which will cost the least.