Instructions: Solve any two of the following problems. Turn in neatly written solutions. If you do any of the programming problems, also send me your code by email. The solutions to problems 1 and 2 can no doubt be found in linear algebra texts; you can discuss these problems with one another, but do not look up the proofs.

1. Prove the following results about eigenvalues and eigenvectors:
   (a) If $A \in \mathbb{R}^{n \times n}$ is nonsingular, then $\lambda = 0$ is not an eigenvalue of $A$.
   (b) If $A \in \mathbb{R}^{n \times n}$ is nonsingular, then the eigenvalues of $A^{-1}$ are the reciprocals of the eigenvalues of $A$.
   (c) If $A \in \mathbb{R}^{n \times n}$, then the eigenvalues of $A^T$ are the same as the eigenvalues of $A$.
   (d) If $A, B \in \mathbb{R}^{n \times n}$ and at least one of $A$ or $B$ is nonsingular, then $AB$ and $BA$ have the same eigenvalues.

2. Suppose $A \in \mathbb{R}^{m \times n}$. The purpose of this problem is to prove that there exist orthogonal matrices $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ and a diagonal matrix $\Sigma \in \mathbb{R}^{m \times n}$ with nonnegative diagonal entries such that $A = U \Sigma V^T$. This is called the singular value decomposition of $A$.
   (a) In this part and the next two, assume that $m \geq n$. Prove that there exists an orthogonal matrix $V \in \mathbb{R}^{n \times n}$ and a diagonal matrix $D \in \mathbb{R}^{n \times n}$ such that $A^T A = V D V^T$. Moreover, prove that the diagonal entries of $D$ are nonnegative. Explain why $D$ and $V$ can be chosen so that its diagonal entries are ordered from largest ($D_{11}$) to smallest ($D_{nn}$).
   (b) By the previous part, we can choose $D$ and $V$ so that the diagonal entries of $D$ are $\sigma^2_1, \ldots, \sigma^2_n$, where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$. Suppose that $\sigma_i > 0$ for $i = 1, 2, \ldots, r$ and $\sigma_i = 0$ for $i = r + 1, r + 2, \ldots, n$. (Note that $r = n$ is a possibility.) Define $u_i = \sigma_i^{-1} A v_i$ for $i = 1, 2, \ldots, r$, where $v_1, \ldots, v_n \in \mathbb{R}^n$ are the columns of $V$. Prove that $\{u_1, u_2, \ldots, u_r\}$ is an orthonormal set in $\mathbb{R}^m$.
   (c) Let $u_{r+1}, u_{r+2}, \ldots, u_m$ be any vectors in $\mathbb{R}^m$ such that $\{u_1, \ldots, u_r, u_{r+1}, \ldots, u_m\}$ is an orthonormal basis for $\mathbb{R}^m$. Then $U = [u_1|u_2|\cdots|u_m] \in \mathbb{R}^{m \times m}$ is an orthogonal matrix. Prove that $A = U \Sigma V^T$, where $\Sigma$ is the diagonal matrix with $\Sigma_{ii} = \sigma_i$, $i = 1, 2, \ldots, n$. (Hint: It suffices to prove that $A v_i = U \Sigma V^T v_i$ for $i = 1, 2, \ldots, n$, since $\{v_1, \ldots, v_n\}$ is a basis for $\mathbb{R}^n$.)
   (d) Prove that if $A \in \mathbb{R}^{m \times n}$ with $m < n$, then $A$ has a singular value decomposition. (Hint: Apply your result for $m \geq n$ to $A^T$.)
3. (a) Write a program implementing the power method to estimate the largest eigenvalue of a matrix $A \in \mathbb{R}^{n \times n}$.

(b) Let the eigenvalues of $A \in \mathbb{R}^{n \times n}$ satisfy

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_k| \geq 0.$$ 

Find examples of $A \in \mathbb{R}^{10 \times 10}$ satisfying each of the following properties and apply your power method program to these matrices. How many iterations does it take in each case to estimate the largest eigenvalue to within a relative error of $10^{-4}$?

i. $|\lambda_2|/|\lambda_1| = 0.01$

ii. $|\lambda_2|/|\lambda_1| = 0.1$

iii. $|\lambda_2|/|\lambda_1| = 0.9$

iv. $|\lambda_2|/|\lambda_1| = 0.99$

4. Write a program implementing the inverse shifted power method to estimate the eigenvalue of $A$ closest to a given scalar $\sigma \in \mathbb{R}$. Demonstrate that your program is correct by executing it on several matrices of size $10 \times 10$.

5. Suppose $A \in \mathbb{R}^{n \times n}$ has real eigenvalues and $\lambda_1$ is one of them. Let $x_1$ be an eigenvector corresponding to $\lambda_1$ and assume that $\|x_1\|_2 = 1$. The following procedure is called deflation; it produces an $(n-1) \times (n-1)$ matrix $B$ which has the same eigenvalues as $A$, except for $\lambda_1$.

(a) Let $\{x_1, x_2, \ldots, x_n\}$ be an orthonormal basis for $\mathbb{R}^n$ and define $X = [x_1 | x_2 | \cdots | x_n]$.

(b) Define $\hat{B} = X^T A X$. Then

$$\hat{B} = \begin{bmatrix} \lambda_1 & v^T \\ 0 & B \end{bmatrix},$$

where $v \in \mathbb{R}^{n-1}$ and $B \in \mathbb{R}^{(n-1) \times (n-1)}$. The matrix $B$ has the same eigenvalues as $A$, except for $\lambda_1$.

We can use deflation, in conjunction with the power method, to find the eigenvalues of $A$, at least when the eigenvalues of $A$ satisfy $|\lambda_1| > |\lambda_2| > \ldots > |\lambda_k|$. (Warning: Numerical stability is a problem with this approach, but that is not the subject of this problem.) Try the above method out on the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ -1 & 3 & 1 \end{bmatrix}.$$ 

Estimate all three eigenvalues in this fashion. You do not have to write a program; you can just do the calculations interactively in MATLAB and turn in a diary of your session (see “help diary” in MATLAB to learn how to create a diary file).

Hint: The one difficulty you might have is in extending an eigenvector such as $x_1$ to an orthonormal basis for $\mathbb{R}^n$. Here is one method: Create a matrix $T$ whose first column is $x_1$ and whose remaining columns contain random numbers. Compute the QR factorization of $T$; then the columns of $Q$ form an orthonormal basis for $\mathbb{R}^n$ and the first column of $Q$ lies in the direction of $x_1$. (There are much more efficient ways to accomplish the same thing, but this is simple.)