From From $t$-wise balanced designs to orthogonal arrays

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(s, v; λ)-Difference matrix based on the group G:

\[ \{a_i - b_i\} = \lambda G, \ G \text{ Abelian.} \]
\[ \{a_i b_i^{-1}\} = \lambda G, \ G \text{ non-Abelian} \]
\[ |G| = s. \]
Example: $(3, 3; 1)$-difference matrix based on $\mathbb{Z}_3$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$
$\text{OA}_\lambda(s, v)$ orthogonal Array:

$v \begin{cases} b_i \\ a_i \end{cases}$

$s^2 \lambda$

$\{(a_i, b_i)\} = \lambda(X \times X)$

$X$ a $s$-element set.
Example: $OA_1(3, 3)$

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\[ \begin{array}{llllll}
D+0 & D+1 & D+2 \\
\end{array} \]

2-c
(s, v; λ)—Difference matrix $\Rightarrow$ OA$_\lambda$(s, v + 1)
Problem: Given $s$ and $\lambda$ what is the maximal $v$ for which a $(s, v; \lambda)$-difference matrix exists?

<table>
<thead>
<tr>
<th>$\lambda \backslash s$</th>
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Data taken from forthcoming book on orthogonal arrays by Hedayat, Sloane and Stufken.
An $\ell$ by $w$ PBD($v$, $\lambda$) is a pair $(X, A)$ where

- $X$ is a $v$-element set of *points*;
- $A$ is a $w$ by $\ell$ array of subsets of $X$ called *blocks*;
- every pair of points is in $\lambda$ blocks; and
- the columns of $A$ are partitions of $X$.

```plaintext
1

5

4

3
```

```
\[\begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}\]```
An example: A 4 by 3 PBD(5,1)
\[ \begin{array}{cccc}
0 & 1 & 2 & 2 \\
0 & 2 & 1 & 1 \\
2 & 2 & 2 & 0 \\
2 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 \\
\end{array} \]

\[ 2M \]
\[ \mathbb{F}_q = \{0, 1, x_2, ..., x_{q-1}\} \]

\[ \ell - \lambda \text{ unequal} \]

\[ \lambda \text{ equal} \]

\[ w < q \]

\[ \text{PBD}(v, \ell) \]

\[ \begin{array}{ccc}
  x_2a & x_2b & \cdots \\
  x_{q-1}a & x_{q-1}b & \text{zeros}
\end{array} \]

\[ \begin{array}{ccc}
  a & \cdots & \cdots \\
  b & \cdots & \cdots \\
  c & \cdots & \cdots
\end{array} \]

\[ \begin{array}{c}
  M \\
  x_2M \\
  x_{q-1}M
\end{array} \]

nonzero difference \( \ell - \lambda \) times
zero difference \((q-1)\lambda\) times
\[ \ell - \lambda - (q-1)\lambda \text{ columns of zeros} \]
**Theorem 1** Let $q$ be a prime power.

If there is an $\ell$ by $w$ PBD($v, \lambda$) with $w \leq q \leq \lfloor \frac{\ell}{\lambda} \rfloor$, then there exists a $(q, v; \ell - \lambda)$-difference matrix.
Theorem

**Theorem 2.** If an OA\((n, k; \lambda)\) exist with 
\(\lambda\) constant rows, then over any group \(G\) of order \(n+1\), 
a \((n + 1, k; \lambda(n - 1))\)-difference matrix exists.

*Example.* \(G = \mathbb{Z}_4\)

\[
\begin{array}{cccccccccc}
1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\
1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\
1 & 2 & 3 & 3 & 1 & 2 & 2 & 3 & 1 \\
1 & 2 & 3 & 2 & 3 & 1 & 3 & 1 & 2 \\
\end{array}
\]

\[
\begin{array}{c|cccccccccc}
a & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\
b & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\
a-b & 0 & 3 & 2 & 1 & 0 & 3 & 2 & 1 & 0 \\
\end{array}
\]

\(\text{OA}_1(3, 4) \Rightarrow (4, 4; 2)-\text{diff. matrix} \Rightarrow \text{OA}_2(4, 5)\)
Problem: Given $s$ and $\lambda$ what is the maximal $v$ for which a $(s, v, \lambda)$–difference matrix exists?

### Upper and lower bounds on $v$

<table>
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<tr>
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<td>60, 11</td>
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Data taken from forthcoming book on orthogonal arrays by Hedayat, Sloane and Stufken.
What’s really going on?

1. We start with a matrix $M$ with entries in $\mathbb{F}_q$
2. Multiply by the nonzeros
   \[ X \mapsto \alpha X \]
3. and then translate by all the elements of $\mathbb{F}_q$
   \[ X \mapsto \alpha X + \beta \]

But this is the affine group
\[ \{ X \mapsto \alpha X + \beta, \text{ where } \alpha, \beta \in \mathbb{F}_q, \alpha \neq 0 \} \]
What’s really going on?

1. We start with a matrix with entries in $v \otimes w$

2. Multiply by the nonzeros $a' \otimes a''$

3. and then translate by all the elements of $v \otimes w$

But this is the affine group $y \otimes x \mod \ell$, where $v \otimes w$.

This group has two orbits on ordered pairs namely:

(1) $\{(x, x) : x \in \mathbb{F}_q\}$

(2) $\{(x, y) : x, y \in \mathbb{F}_q, x \neq y\}$

9-a
Orthogonal array of size $N$, degree $k$, order $s$ and strength 3: $OA_\lambda(3, k, s)$

$$N = s^3 \lambda$$

Every $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ occurs $\lambda$ times

Symbols from an $s$-set.
Or

Example: An $OA_1(3, 4, 3)$

00000000011111111122222222
012012012012012012012012012
012120201120201012201012120
012201120120012201201120012

10-a
Problem: Given $s$ and $\lambda$ what is the maximal $k$ for which a $OA_\lambda(3, k, s)$ exists?

Existence results for orthogonal arrays of strength more than 2.


1. **Bose-Bush (1952):** $k \leq \left\lceil \frac{\lambda s^2 - 1}{s - 1} \right\rceil + 1$

2. An $OA_\lambda(t, k, s)$ is a $OA_{s\lambda}(t - 1, k, s)$.

3. A $OA_\lambda(t - 1, k - 1, s)$ can be obtained from a $OA_\lambda(t, k, s)$ by first selecting all columns that contain a fix symbol $x$ in a given row $i$ and then deleting row $i$.

4. **Bush (1952):** If $s > t$ is a prime power, then an $OA_1(t, s + 1, s)$ exists.

5. **Bush (1952):** If $s > 3$ is a power of 2, then an $OA_1(3, s + 2, s)$ exists.
6. Bush (1952): If \( OA_{\lambda_1}(t, k, s_1) \) and \( OA_{\lambda_2}(t, k, s_2) \) exists, then an \( OA_{\lambda_1\lambda_2}(t, k, s_1s_2) \) also exists.

7. Atsumi (1983): If \( t \) is even, then any \( OA_{\lambda}(t, k, 2) \) implies the existence of an \( OA_{\lambda}(t + 1, k + 1, 2) \). In particular an \( OA_n(3, 4n, 2) \) exists whenever there is a Hadamard matrix of order \( 4n \).

   If \( m, n \geq 0 \) are integers and \( p \) is a prime, then a \( OA_{p^{(t-1)}}(t, p^{m+n} + 1, p^m) \) exists for all \( t \geq 3 \) and an \( OA_{p^{(t-2)-m}}(t, pm + n, p^m) \) exists for all \( t > p^n \).

   If \( p \) is a prime and an \( OA_{\lambda}(3, k, p^m) \) exists, then a \( OA_{\lambda p^{2(m+n)}}(3, kp^{m+n}, p^m) \) exists for all integers \( n \geq 0 \).

10. Mukhopadhyay (1981): If an \( OA_{\lambda}(3, r, s) \) exists, so does an \( OA_{s\lambda}(3, 2r, s) \).
A resolvable $3-(wk, k, \lambda)$ design is a pair $(X, A)$ where

- $X$ is a $wk$-element set of points;
- $A$ is a $\ell$ by $w$ array of $k$-subsets of $X$ called blocks;
- every pair of points is in $\lambda$ blocks; and
- the rows of $A$ are partitions of $X$.

Example: A resolvable $3-(8, 4, 1)$

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<td>5712</td>
<td>6123</td>
<td>7234</td>
<td>1345</td>
<td>2456</td>
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</table>
A resolvable $h$-design is a pair $\{d, e\}$ where
d is a \(\mathcal{H}\)-element set of
points;
e is a \(\mathcal{B}\)-by\(\mathcal{H}\) array of
\(\mathcal{H}\)-subsets of
\(\mathcal{A}\) called
blocks;
every pair of points is in
a block; and
d the rows of
\(\mathcal{C}\) are partitions of
\(\mathcal{A}\).

Example: A resolvable 3-($9, 3, 1$)

\[
\begin{array}{cccccccccc}
789 & 781 & 782 & 783 & 784 & 785 & 786 & 715 & 726 & 739 & 741 & 752 & 763 & 794 \\
124 & 235 & 346 & 459 & 561 & 692 & 913 & 826 & 839 & 841 & 852 & 863 & 894 & 815 \\
563 & 694 & 915 & 126 & 239 & 341 & 452 & 943 & 154 & 265 & 396 & 419 & 521 & 632 \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
723 & 734 & 745 & 756 & 769 & 791 & 712 & 746 & 759 & 761 & 792 & 713 & 724 & 735 \\
845 & 856 & 869 & 891 & 812 & 823 & 834 & 925 & 136 & 249 & 351 & 462 & 593 & 614 \\
916 & 129 & 231 & 342 & 453 & 564 & 695 & 813 & 824 & 835 & 846 & 859 & 861 & 892 \\
\end{array}
\]
Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_{n+1}\}$, with $n + 1 \geq w$. 

\[
\begin{array}{cccccccccccc}
\omega_1 & 789 & 781 & 782 & 783 & 784 & 785 & 786 & 715 & 726 & 739 \\
\omega_2 & 124 & 235 & 346 & 459 & 561 & 692 & 913 & 826 & 839 & 841 & \cdots \\
\omega_3 & 563 & 694 & 915 & 126 & 239 & 341 & 452 & 943 & 154 & 265 \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
1 & \omega_2 & \omega_1 & \omega_3 & \omega_3 & \omega_2 & \omega_3 & \omega_2 & \omega_1 & \omega_3 & \omega_2 \\
2 & \omega_2 & \omega_2 & \omega_1 & \omega_3 & \omega_3 & \omega_2 & \omega_3 & \omega_2 & \omega_1 & \omega_3 \\
3 & \omega_3 & \omega_2 & \omega_2 & \omega_1 & \omega_3 & \omega_2 & \omega_3 & \omega_2 & \omega_1 & \\
4 & \omega_2 & \omega_3 & \omega_2 & \omega_2 & \omega_1 & \omega_3 & \omega_3 & \omega_3 & \omega_3 & \omega_2 & \cdots \\
5 & \omega_3 & \omega_2 & \omega_3 & \omega_2 & \omega_2 & \omega_1 & \omega_3 & \omega_1 & \omega_3 & \omega_3 \\
6 & \omega_3 & \omega_3 & \omega_2 & \omega_3 & \omega_2 & \omega_2 & \omega_1 & \omega_2 & \omega_1 & \omega_3 \\
7 & \omega_1 & \omega_1 & \omega_1 & \omega_1 & \omega_1 & \omega_1 & \omega_1 & \omega_1 & \omega_1 & \omega_1 \\
8 & \omega_1 & \omega_1 & \omega_1 & \omega_1 & \omega_1 & \omega_1 & \omega_1 & \omega_2 & \omega_2 & \omega_2 \\
9 & \omega_1 & \omega_3 & \omega_3 & \omega_2 & \omega_3 & \omega_2 & \omega_3 & \omega_2 & \omega_1 & \\
\end{array}
\]

$M$
Let $G$ be a group acting 3-transitively on $\Omega$

\[ G = \{g_1, g_2, \ldots, g_{|G|}\} \]

\[ |G| = m(n + 1)n(n - 1) = m(n^3 - n). \]

\[ M^G = [M^{g_1}, M^{g_2}, \ldots, M^{g_{|G|}}] \]

$C = \text{each constant column repeated } x \text{ times}$. 
($v$ by $x(n + 1)$ matrix)

The number of blocks containing a pair of points disjoint from a third is

\[ b_2^1 = \binom{v-3}{k-2} \lambda / \binom{v-t}{k-1}. \]

Claim $[M^G, C]$ is an orthogonal array.
Consider any three rows

<table>
<thead>
<tr>
<th>Type</th>
<th>in $M$</th>
<th>in $M^G$</th>
<th>in $C$</th>
<th>Total</th>
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<tbody>
<tr>
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<td></td>
<td>$\lambda$</td>
<td>$\frac{</td>
<td>G</td>
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<tr>
<td>$x$</td>
<td>$b_2^1$</td>
<td>$\frac{</td>
<td>G</td>
<td>}{(n+1)n}b_2^1$</td>
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<tr>
<td>$y$</td>
<td>$b_2^1$</td>
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<td>G</td>
<td>}{(n+1)\lambda}b_2^1$</td>
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<td>$x$</td>
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<td>$x$</td>
<td>$b_2^1$</td>
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<td>G</td>
<td>}{(n+1)n}b_2^1$</td>
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<tr>
<td>$y$</td>
<td>$r - \lambda - 3b_2^1$</td>
<td>$\frac{</td>
<td>G</td>
<td>(r-\lambda-3b_2^1)}{(n+1)n(n-1)}$</td>
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</tbody>
</table>
Choose $n$ and $x$ so that

1. $m(r - \lambda - 3b_2^1) = mb_2^1(n - 1)$
2. $n + 1 \geq w$.
3. $x + mn(n - 1)\lambda = mb_2^1(n - 1)$
Choose

**Theorem.** Let $G$ act 3-transitively on the $(n + 1)$-set $\Omega$ and let $m(n^3 - n)$ be the order of $G$. If a resolvable 3-$(wk, k, \lambda)$ design $(X, B)$ exists such that

1. $n = (r - \lambda)/b^1_2 - 2$
2. $n + 1 \geq w$
3. $n \leq b^1_2/\lambda$,

then an $OA_{mb^1_2(n-1)}(3, wk, n + 1)$ also exists.
Use $G = PGL(2, q)$ on $\Omega = \mathbb{F}(q) \cup \{\infty\}$ and resolvable SQS(3q+5)’s.

**Corollary A.**
Let $q$ be a prime power, $q \equiv 1, 5, 9 \pmod{12}$. Then there exists an

$$OA_{(3q+1)(q-1)/2}(3, 3q + 5, q + 1),$$

except possibly for $q = 197$ or 773.
Use $G = \text{PGL}(2, q)$ on $\Omega = \mathbb{F}(q) \cup \{\infty\}$ and resolvable $3$-$(2q + 4, 3, 1)$.

**Corollary B.**

Let $q$ be a prime power, $q \equiv 1 \pmod{3}$. Then there exists an

$$OA(2q+1)(q-1)(3, 2q + 4, q + 1).$$
We don't need 3-designs!

All that is required is a resolvable set-system \((X, B)\) such that

1. The number of blocks containing three points \(x, y, z\) is a constant \(\lambda\) that does not depend on the choice of \(x, y, z\)

2. The number of blocks containing two points \(x, y\) and disjoint from a third point \(z \in X\), is a constant \(b^1_2\) that does not depend on the choice of \(x, y, z\)
We don't need 3-designs!

All that is required is a resolvable set-system \( F \) such that

1. The number of blocks containing three points is a constant \( D \) that does not depend on the choice of \( o \).

2. The number of blocks containing two points and disjoint from a third point is a constant \( \tau \) that does not depend on the choice of \( o \).

Do they exist? Can they be resolvable?
**Example:** The 1-factorization of the complete graph is a resolvable near 3-design with

$$\lambda = 0 \quad \text{and} \quad b_2^1 = 1$$
Example:

The revised theorem is:

**Theorem:** Let \( G \) act 3-transitively on the \((n + 1)\)-element set \( \Omega \) and let \( m(n^3 - n) \) be the order of \( G \). If a resolvable near 3-design \((X, \mathcal{B})\) exists such that 
\[
n = \frac{(r - \lambda)}{b_2^1} - 2 \text{ with } w - 1 \leq n \leq \frac{b_2^1}{\lambda},
\]
then a resolvable \( OA_{m b_2^1(n-1)}(3, |X|, n + 1) \) also exists.
Example:

**Corollary:** Let \( q \) be an odd prime power. Then there exists an \( OA_{q-1}(3, q + 3, q + 1) \).

Proof: Use 1-factorization of \( K_{q+3} \) and \( GL_2(q) \).
The same methods can be used to construct covering arrays.

A Covering array of size $N$, degree $k$, order $v$ and strength 3: $CA(N, 3, k, v)$

Symbols from an $v$-set.

Example: Use a 1-factorization of $K_6$ and the group $S_3$ to obtain an optimal covering array: $CA(33, 3, 6, 3)$.

```
0 1 2 2 1 1 2 0 0 2 2 0 1 1 0 0 2 2 1 1 2 2 0 2 2 0 1 1 0 1 2 0 1 2 0 1 0 2 2 0 1 2 1 0 0 1 2 0 1 2
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1 0 1 2 2 2 1 2 0 0 0 2 0 1 1 2 0 2 1 1 0 1 0 2 2 1 2 1 0 0 0 1 2
0 0 0 0 0 1 1 1 1 1 2 2 2 2 2 0 0 0 0 0 1 1 1 1 2 2 2 2 0 1 2
```
We also obtain a $CA(88, 3, 8, 4)$ from a 1-factorization of $K_8$ and the group $A_4$. This is best known, but may not be optimal.

This is work in progress with: M.A. Chateauneuf and C.J. Colbourn
1. What is combinatorial mathematics? Combinatorial mathematics also referred to as combinatorial analysis or combinatorics, is a mathematical discipline that began in ancient times. According to legend the Chinese Emperor Yu (c. 2200 B.C.) observed the magic square

\[
\begin{bmatrix}
8 & 1 & 6 \\
3 & 5 & 7 \\
4 & 9 & 2
\end{bmatrix}
\]

on the shell of a divine turtle. ...

\[
\begin{bmatrix}
8 & 1 & 6 \\
3 & 5 & 7 \\
4 & 9 & 2 \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{bmatrix}
+ 
\begin{bmatrix}
1 & 0 & 2 \\
2 & 1 & 0 \\
0 & 2 & 1 \\
\end{bmatrix}
+ 3 
\begin{bmatrix}
2 & 0 & 1 \\
0 & 1 & 2 \\
1 & 2 & 0 \\
\end{bmatrix}
\]