Chapter 3 - Lecture 4
Moments and Moment Generating Functions

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Moments
  Moments
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  Skewness

Moment Generating Function
  Definition
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  Proposition

Exercises
Definition

- The expected value of a power of a random variable is called **moment** (or **moment around 0**).
- **First moment** or Expected value $E(X)$
- **Second moment** $E(X^2)$
- **Third moment** $E(X^3)$ and so on...
Central Moments

- **Central moments** are the expected value of the difference of a random variable and its expected value (or mean) to a power. It is also called **moment about the mean**.
- **Second central moment** or variance $E \left( (X - E(X))^2 \right)$
- **Third central moment** $E \left( (X - E(X))^3 \right)$ and so on...
Moments and central moments are useful because they give us useful information regarding the distribution of a random variable.

Until now we know how to find the mean and the variance using moments and central moments. Another useful measure is skewness (see Chapter 1).
Definition

- **Skewness** measures the departure from symmetry
- Using the third and second central moments we can calculate skewness:

\[
E \left( \frac{(X - E(X))^3}{\sigma^3} \right)
\]
Example

- In State College there are 10,000 families. I asked them how many kids they have and I get the following answers:
  - 0 kids: 900 families
  - 1 kid: 1700 families
  - 2 kids: 4000 families
  - 3 kids: 2100 families
  - 4 kids: 700 families
  - 5 kids: 400 families
  - 6 kids: 200 families

- Find the skewness of $X =$ number of kids in a family in SC.
Is not always easy to calculate moments and central moments. That's why we use moment generating functions.

The **moment generating function (mgf)** of a discrete random variable $X$ is defined to be

$$M_X(t) = E\left(e^{tX}\right) = \sum_{x \in D} e^{tx} p(x)$$
Bernoulli random variable

- Find the moment generating function of a Bernoulli random variable
Proposition

- If the mgf exists and is the same for two distributions, then the two distributions are the same.

- In other words, the moment generating function uniquely specifies the probability distribution.
Example

In State College there are 10000 families. I asked them how many kids they have and I get the following answers:

- 0 kids: 900 families
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Find the mgf of $X =$ number of kids in a family in SC.
Example

If a variable $Y$ has mgf the following find its distribution function:

$$M_Y(t) = 0.09 + 0.17e^t + 0.4e^{2t} + 0.21e^{3t} + 0.07e^{4t} + 0.04e^{5t} + 0.02e^{6t}$$
Theorem

- If the mgf exists then

\[ E(X^r) = M_X^{(r)}(0) \]

- Proof?
Example

- Let's go back to the example with the kids per family in State College. Use the Theorem in the previous page to find $E(X)$, $E(X^2)$, $E(X^3)$ and $\text{var}(X)$. 
Proposition

- If $X$ has mgf $M_X(t)$ and $Y = aX + b$ then $M_Y(t) = e^{bt} M_X(at)$
- Proof?
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Exercises 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 56, 57