Practice Exam Problems of Group Theory - Solutions

Abstract Algebra 3310

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Problem i. Let $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}^+$ be a group homomorphism (where $\mathbb{R}^+$ as usual indicates the multiplicative group of positive real numbers), such that $\phi(0,1) = 4$ and $\phi(1,0) = 2$. Determine the kernel of ϕ.

Solution. For every $m, n \in \mathbb{Z}$, we have

$$\phi(n,m) = \phi(n(1,0) + m(0,1)) = (\phi(1,0))^n \cdot (\phi(0,1))^m = 2^n \cdot 4^m = 2^{n+2m}.$$

Hence, $(n,m) \in \mathbb{Z} \times \mathbb{Z}$ is in $\ker(\phi)$ if and only $\phi(n,m) = 1 = 2^0$, if and only $n = -2m$.

In other words, the elements of $\mathbb{Z} \times \mathbb{Z}$ in the kernel of ϕ are those of the form $(-2m, m)$, where m ranges over the integers.

Problem ii. Show that there exists no group homomorphism $\phi : \mathbb{Z}_n \to \mathbb{Z}$ such that $\phi(1) > 0$.

Solution. Let $\phi : \mathbb{Z}_n \to \mathbb{Z}$ be any morphism. Since $\mathbb{Z}_n$ is finite, the image of ϕ must have finite order. But all the subgroups of $\mathbb{Z}$ are the cyclic groups (n), and the only one which is finite is (0). Thus $\phi(a) = 0$ for all $a \in \mathbb{Z}_n$, and in particular $\phi(1) = 0$.

Problem iii. Let $G = 10\mathbb{Z}$ (the additive group of all integer multiples of 10), and $H = 20\mathbb{Z}$. (Notice that H is a normal subgroup of G.) Show that G/H is isomorphic to $\mathbb{Z}_2$.

Solution. Since all groups of order 2 are isomorphic to $\mathbb{Z}_2$, we only need to show that $T := G/H$ has 2 elements. Notice that all multiples of 10 are either congruent to 0 or to 10 modulo 20, i.e., they are all equal to $20m$ or to $20m + 10$, for some integer m.

But, for every m, the class of $20m$ in T is the same as that of 0, since $(20m)-0 = 20m \in H$. Similarly, the class of $20m+10$ is the same as that of 10, since $(20m+10)-10 = 20m \in H$.

Therefore there are only 2 elements in T, namely $\overline{0}$ and $\overline{10}$, as desired.

Problem iiii. Let $\phi : \mathbb{Z}_{1000} \to \mathbb{Z}_2$ be a surjective homomorphism. Determine $\phi(1)$.

Solution. $\phi(1) \in \mathbb{Z}_2 = \{0,1\}$. If $\phi(1) = 0$, then, for every $a \in \mathbb{Z}_{1000}$, we have

$$\phi(a) = a \cdot \phi(1) = a \cdot 0 = 0,$$

against the fact that ϕ is surjective. Hence $\phi(1) = 1$.

Problem v. Let G be any (not necessarily commutative) group and H a normal subgroup of G. Show that, for any choice of $h \in H$, $g \in G$, and $m, n \in \mathbb{Z}$, we have

$$g^2 h^m g^{-4} h^n g^2 \in H.$$

Solution. Notice that, because H is a group, h^m and h^n are in H for any integers m and n. Thus, since H is normal, both $gh^m g^{-1}$ and $g^{-1}h^n(g^{-1})^{-1} = g^{-1}h^n g$ belong to H. Therefore their product, that is $gh^m g^{-2} h^n g$, is in H as well, as desired.

Problem vi. Let ϕ be a map from $\mathbb{R}$ (the additive group of all real numbers) to $\mathbb{R} \setminus \{0\}$ (the multiplicative group of all non-zero real numbers), defined by

$$\phi(x) = 2^x.$$

i) Prove that ϕ is a homomorphism.
ii) Is ϕ an isomorphism?

Solution. i) For any $x, y \in \mathbb{R}$, we have

$$\phi(x+y) = 2^{x+y} = 2^x \cdot 2^y = \phi(x) \cdot \phi(y).$$

ii) No. The image of ϕ is not all of $\mathbb{R} \setminus \{0\}$ (since $2^x > 0$ for all $x \in \mathbb{R}$), and therefore ϕ is not surjective.

Problem vii. Let G be a group, and suppose that, for any $a, b \in G$, we have $a^2b^2 = (ab)^2$. Prove that G is abelian.

Solution. We want to show that, for any $a, b \in G$, $ab = ba$.

Since $(ab)^2 = abab$, and $a^2b^2 = aabb$, by hypothesis we have that $aabb = abab$.

Let us multiply both sides by b^{-1} on the right and by a^{-1} on the left. Using the associative property, we get $(a^{-1}a)ab(bb^{-1}) = (a^{-1}a)ba(bb^{-1})$, i.e. (if we call e the identity of G) $eabe = ebae$, whence $ab = ba$, as we wanted to show.