One-to-one functions and inverse functions

What does f^{-1} do?
The theme for this page is

$$f^{-1} \text{ UNdoes whatever } f \text{ does.}$$

1. Suppose we have a function f which takes a number (the input), triples it, and then adds 5. Write a formula for the function:

   $$f(x) = \rule{3cm}{0.4pt}$$

   (Copy this expression into the blank in exercise 4 below.)
   First we'll deal with things logically, using words to describe what f^{-1} must do.

2. What does the function f^{-1} do? Think about undoing the processes of f (be careful about the order of operations!) and write your answer in a sentence (like the first sentence in the previous question). Don't do any algebra here.

3. Working directly with your answer to the previous exercise, write a formula for the inverse of f:

   $$f^{-1}(x) = \rule{3cm}{0.4pt}$$

Now we'll try to arrive at this same formula algebraically.

4. Take the expression $y = f(x)$:

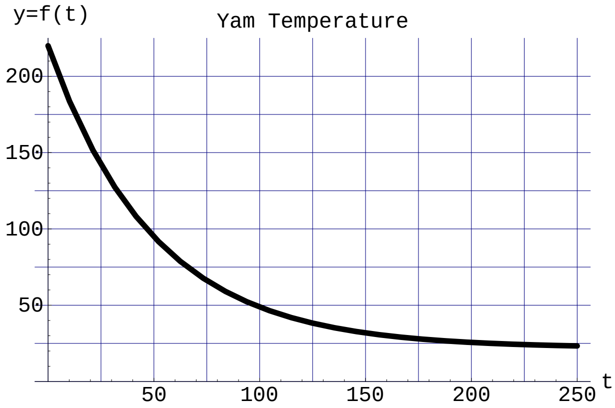
   $$y = \rule{3cm}{0.4pt}$$

   and solve it for x in terms of y (do NOT interchange the two variables!).

5. Why does this tell you a formula for $f^{-1}(y)$? (Remember: $y = f(x)$ is equivalent to $x = f^{-1}(y)$.)
6. Does your formula for f^{-1} in part 4 match the formula from part 1? If not, fix them now.
7. As a final check, compute $f^{-1}(f(x))$ and $f(f^{-1}(x))$.

Input/output If you're working with a function $f(x)$, then it's easy to answer the question "What is the output of f when I input a value of 7?" (the answer is $f(7)$, of course!). But sometimes you may want to ask the question "What do I need to put INto f to get an OUTPUT of 7?" ...you need a function which inputs the output of f and outputs the input of f. This is f^{-1}. Remember: $b = f(a)$ is the same as $f^{-1}(b) = a$

Suppose that a yam is taken out of the oven and left on the counter, It's temperature y (in ^oC) is a function of the time t (in minutes) that the yam has been on the counter, as shown in the graph below.



8. The input to f is the time t (in minutes) that the yam has been on the counter. What is the output of f? What about f^{-1}?

f: input: *minutes on the counter*
output: ____________________

f^{-1}: input: ____________________
output: ____________________

9. What is $f(30)$ (approximately)? What does this number represent?

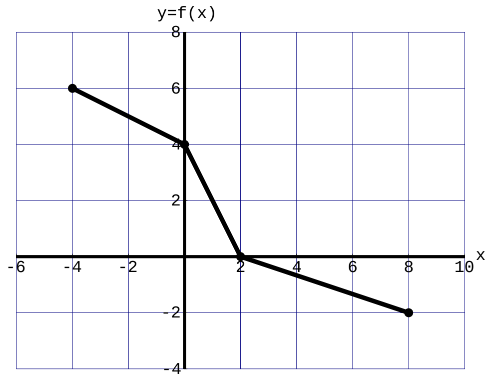
10. What is $f^{-1}(50)$ (approximately)? What does this number represent?

11. Write a sentence which says the same thing as the algebraic statement "$f(30) = 131$."
(Include units for both the 30 and the 131, but don't use any symbols except numerals. Your answer should include the words "temperature", "yam", "time" (or "minutes") and "degrees".)

12. Write a sentence which says the same thing as the algebraic statement "$f^{-1}(100) = 47$."
(Your answer should include the words "temperature", "yam", "time" (or "minutes") and "degrees".)

Inverses from graphs

The first few questions deal with the function $f(x)$ graphed below. (Notice that each box is *two* units wide).



13. Based on the graph, list at least three points on the graph of $y = f(x)$.

14. Based on your previous answer, list at least three points on the graph of $y = f^{-1}(x)$.

15. Sketch a graph of $y = f^{-1}(x)$ on the same axes as $y = f(x)$ above. (The points you listed in your answer to the previous problem should be on your graph.)

16. What are the domain and range of f?

17. What are the domain and range of f^{-1}?

If you have more time:

18. Use your graphing calculator to graph $g(x) = x^3$ and its inverse, $g^{-1}(x) = \sqrt[3]{x}$ on the same graph, with a viewing window of dimensions $-2 \leq x \leq 2$, $-8 \leq y \leq 8$. (You might want to include the graph of $y = x$ there, too.)

    Does it look like the two graphs are mirror images about the line $y = x$? Should they be? What's going on here?

19. TRUE or FALSE:

    (a) If $x^3 = 7^3$, then $x = 7$. **T** **F**

    (b) If $x^2 = 7^2$, then $x = 7$. **T** **F**

    What does this have to do with inverse functions?

20. TRUE or FALSE:

    (a) The inverse of $f(x) = x^2$ is $f^{-1}(x) = \sqrt{x}$. **T** **F**

    (b) $f(x) = x^2$ does not have an inverse function because it is not one-to-one. **T** **F**

    (c) The function $f(x) = x^2$ does not pass the horizontal line test. **T** **F**

    How are these answers related?

If you have more time:

21. Do you remember this function from Worksheet #4?

$$f(x) = \frac{x-3}{x-5}$$

    (a) Find f^{-1}, the inverse of this function, if it exists.

    (b) What is the domain of f^{-1}?

    (c) What does this tell us about f itself?

    (d) What is the range of f^{-1}

22. Go back to Worksheet #4 (functions), and decide which of those functions are one-to-one. What is the meaning of the inverse function in each case?

For Instructors:

SUGGESTIONS on how to USE worksheet #7

The students should know a little bit about INVERSES:

- $f(a) = b$ means the same thing as $f^{-1}(b) = a$.
- f^{-1} switches input/output of f.
- Inverse exists if f is one-to-one (horiz. line test)
- Graphs of f and f^{-1} are mirror images about $y = x$ (if the scales are the same on both axes).
- (a, b) is on the graph of $y = f(x)$ means that (b, a) is on the graph of $y = f^{-1}(x)$, and vice-versa.
- How to compute $f^{-1}(x)$ algebraically by solving for x, and how to check the computation by finding $f(f^{-1}(x))$ and $f^{-1}(f(x))$.