Effective Yield Behavior for Mixtures of Elastic/Plastic Materials*

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A method is described for bounding the effective yield behavior of a mixture composed of anisotropic elastic/plastic materials. It is assumed that the only mechanism for deformation is plastic deformation of individual grains and that the set of stresses causing plastic flow (the "yield surface") is known for each constituent material.

Variational principles are used to generate an upper bound on the effective yield surface from bounds on the moduli of a suitably chosen *linearly elastic* medium with the same geometry as the given material. This work is then compared to that of Taylor and Bishop & Hill, who derived an upper bound for polycrystals deforming by slip. Their results are shown to be recovered when using the elementary arithmetic-mean bound for the linear material. Moreover, it is shown that if one uses more sophisticated bounds on linear properties then the new bound is tighter than that of Taylor and Bishop & Hill. An added advantage of the proposed bounding procedure is the ability to obtain bounds which depend on two-point correlation functions by using the analogous linear bounds.

Our heterogeneous medium is a mixture of materials, each having its own yield function determining when plastic flow occurs. If we call the set of allowable stresses "P", then the boundary of P ("∂P") represents the yield surface and consists of stresses at which plastic flow begins. Locally (i.e., in each material), the flow rule holds:

$$\begin{array}{rcl} \sigma \in \text{interior of } P & \Rightarrow & D = 0 \\ \sigma \in \partial P & \Rightarrow & D \perp \partial P \text{ at } \sigma \end{array} \tag{1}$$

D represents the *plastic* strain-rate.

In addition, we assume that there is no slipping along the boundaries between two different materials and that the mixture is in equilibrium:

$$D(\vec{v}) = \frac{1}{2}(\nabla \vec{v} + \nabla \vec{v}^T) \quad \text{with continuous velocity } \vec{v} \tag{2}$$

$$\nabla \cdot \sigma = 0. \tag{3}$$

In analogy with the homogeneous material, the relationship between the "macroscopic" stress and strain rate can be described by a yield surface $Y^*(\Sigma)$. We define the macroscopic stress Σ^0 and the macroscopic strain rate D^0 to be averages over the unit volume Ω occupied by our mixture.

For simplicity of notation, in the following all microscopic variables are assumed to be functions of position (e.g., σ, D, Y, P) and all integrals are taken over physical space ($\vec{x} \in \Omega$) unless noted otherwise.

We look for bounds of the form

$$D^0 \cdot \Sigma^0 \leq \text{ constant }, \tag{4}$$

which determine a hyperplane in Σ^0-space. A set of such hyperplanes corresponding to various values of D^0 form an envelope containing the stress domain and as such forms an "upper bound."

To derive these new upper bounds, we begin with a variational expression for the energy dissipation of the composite $D^0 \cdot \Sigma^0$.

The actual equilibrium strain rate is such that it minimizes total energy dissipation, given that the stress is related to the strain rate through the flow rule. Furthermore, this stress maximizes $\sigma \cdot D$ over all "allowable" stresses (i.e., over $\sigma \in P$). Combining these observations allows us to express the energy dissipation as

$$D^0 \cdot \Sigma^0 = \min_{D(\vec{v})} \max_{\sigma(\vec{x}) \in P(\vec{x})} \int_\Omega \sigma(\vec{x}) \cdot D(\vec{x})\, d\vec{x}. \tag{5}$$

where $\vec{v}(\vec{x})$ is required to have boundary values of $D^0\vec{x}$.

We derive our new principle from this expression by adding and subtracting the energy of a linear elastic

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comparison medium and performing one of the minimizations explicitly. This gives us the

<u>New Variational Principle</u>

$$(D^0 \cdot \Sigma^0)^2 = \min_{S(\vec{x})>0} \left(D^0 \cdot (S^{-1})^* D^0 \right) \left(\int_\Omega \max_{\sigma \in P(\vec{x})} \sigma \cdot S\sigma \right) \qquad (6)$$

where $S(\vec{x})$ ranges over all positive-definite fourth-order tensors. A complete derivation of this variational principle may be found in [4], and similar variational principles for vonMises-type materials appear in [5] and [3].

Obtaining bounds from this variational principle involves two choices. One must choose the linear comparison medium, which in practice will be piecewise constant (taking a different value in each phase of the actual material). Also, one must choose a linear bound on $(S^{-1})^*$ from the many available in the literature.

The results obtained by Taylor [6] and Bishop & Hill [1] can be obtained also from the new variational principle. If we choose our comparison medium S so that

$$D^0 = S(\vec{x})\,\sigma(P_x, D^0) \qquad (7)$$

$$\max_{\sigma \in P(x)} \sigma \cdot S(x)\sigma = D^0 \cdot \sigma(P_x, D^0) \qquad (8)$$

and use the arithmetic-mean bound for $(S^{-1})^*$, both factors on the right side of equation (6) are equal to the Taylor bound. Thus if we use this comparison medium and a bound better than the arithmetic mean, we decrease the value of the first factor in (6) and thereby obtain a bound tighter than the Taylor/Bishop-Hill bound.

For a sample calculation, we consider a mixture of two materials, each obeying a vonMises yield criterion:

$$P_i = \{\sigma : \ \|\sigma'\|^2 \le k_i\} \qquad i = 1, 2 \qquad (9)$$

where σ' is the deviatoric part of σ and $k_1 < k_2$. The average grain shape is assumed to be ellipsoidal, with the two phases distributed randomly among the grains. We choose the S tensor to be a scalar multiple of the (deviatoric) identity in each phase, and use Willis' bound on $(S^{-1})^*$ for ellipsoidal geometries [7].

When the only nonzero component of D^0 is ${D^0}_{12}$ (similarly for ${D^0}_{13}$,${D^0}_{23}$), the bound has a simple form:

$$\Sigma^0_{12} \le \frac{f_1 k_1 + f_2\sqrt{(1-s)\{k_2^2 - s(f_1 k_1^2 + f_2 k_2^2)\}}}{1 - f_2 s} \qquad (10)$$

where $f_1 = 1 - f_2$ is the volume fraction of the first phase and $s = S^{Esh}_{1212}$ is a component of Eshelby's tensor [2]. The dependence of this bound on the aspect ratio of ellipsoidal grains with semi-axes $a_1 = a_2 \neq a_3$ is shown in the figure below. It is interesting to note that aspect ratios greater than one (prolate ellipsoids) have very little effect on the bounds. The most dramatic effect is for discs ($a_3 \ll a_1$) and the limiting laminate ($a_3 \to 0$), where the upper bound coincides with the constant-stress lower bound for Σ^0_{13}.

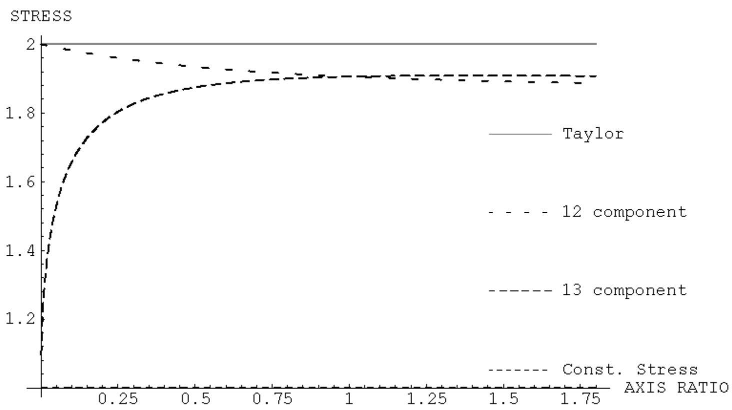

Figure 1: New upper bounds on Σ^0_{12} and Σ^0_{13} with constant-strain-rate upper bound ("Taylor") and constant-stress lower bound ("Reuss") for $k_1 = 1, k_2 = 3,\ f_1 = f_2 = 0.5$.

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