2.2 -- Separable Variables

Separable Equation Form:

\[ \frac{dy}{dx} = g(x) h(y) \]  \hspace{1cm} (1)

- We can use algebra to separate the variables as follows:

\[ \frac{dy}{h(y)} = g(x) \, dx \]  \hspace{1cm} (2)

- We can then integrate both sides of the equation.

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Let's try an example: our temperature sensor model.

\[ \frac{dT}{dt} + 2T = 10 \]

We need to separate the \( T \)'s from the \( t \)'s,

\[ \frac{dT}{dt} = 10 - 2T \]  \hspace{1cm} (1)

\[ dT = (10 - 2T) \, dt \]

\[ \int \frac{dT}{10 - 2T} = \int dt \]  \hspace{1cm} (2)
Lec Section 2.2

\[ \frac{-1}{2} \int \frac{du}{u} = t + C \]

\[ u = 10 - 2T \]

\[ du = -2 \, dT \]

\[ dT = -\frac{du}{2} \]

\[ \frac{1}{2} \ln(10 - 2T) = t + C \]

\[ \ln(10 - 2T) = -2t + C_1 \]

\[ C_1 = -2C \]

\[ 10 - 2T = e^{(2t + C_1)} = e^{-2t} \, e^{C_1} = C_2 e^{-2t} \]

\[ C_2 = e^{C_1} \]

\[ 2T = 10 - C_2 e^{-2t} \]

\[ T = 5 - C_3 e^{-2t} \]

\[ C_3 = C_2/2 \]

If we have an initial value for \( T \) we can solve for the constant of integration \( C_3 \).

Let \( T(0) = 0 \)

\[ 0 = 5 - C_3 \]

\[ C_3 = 5 \]

\[ T = 5 - 5 e^{-2t} \]

\[ \frac{dT}{dt} = 0 \]

\[ T | \quad \frac{dT}{dt} \]

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Problem 2.2.6.
\[ \frac{dy}{dx} + 2xy^2 = 0 \]
\[ \frac{dy}{dx} = -2xy^2 \]
\[ dy = -2xy^2 \, dx \]
\[ \frac{dy}{y^2} = -2x \, dx \]
\[ \int y^{-2} \, dy = -2 \int x \, dx \]
\[ y = -x^2 + c \]

\[ y = \frac{1}{x^2 + c} \]

Problem 2.2.12.
\[ \sin 3x \, dx + 2y \cos^3 3x \, dy = 0 \]
\[ \frac{\sin 3x}{\cos^3 3x} \, dx + 2y \, dy = 0 \]
\[ \tan 3x \sec^2 3x \, dx + 2y \, dx = 0 \]

\[ dv = \sec^2 3x \, dx \quad dv = \tan 3x \, dx \]
\[ u = \frac{1}{3} \tan 3x \quad du = 3\sec^3 3x \]

\[ \int \tan 3x \sec^3 3x \, dx = \frac{1}{3} \tan^4 3x \]
\[ \int \sec^2 3x \tan 3x \, dx = \int \sec 3x \left( \sec 3x \tan 3x \right) \, dx \]

Let \( u = \sec 3x \) \quad du = 3 \sec 3x \tan 3x \, dx \quad dx = \frac{1}{3 \sec 3x \tan 3x} \]

\[ \frac{1}{3} \int u \, du = \frac{1}{6} u^2 + c \]
\[ = \frac{1}{6} \sec^2 3x \]

\[ \frac{\sin 3x}{\cos^3 3x} = \frac{1}{\cos 3x} \cdot \frac{\sin 3x}{\cos 2x} \]
\[ \tan 3x = \sec^2 3x - 1 \quad \tan^2 3x = \sec^2 3x + c \]
\[
\frac{dy}{dx} = \frac{y^2 - 1}{x^2 - 1} \quad y(2) = 2
\]

\[
\frac{dy}{y^2 - 1} = \frac{dx}{x^2 - 1}
\]

\[
\frac{dy}{(y-1)(y+1)} = \frac{dx}{(x-1)(x+1)}
\]

\[
\frac{1}{(y-1)(y+1)} = \frac{A}{y-1} + \frac{B}{y+1}
\]

\[
1 = (y+1)A + (y-1)B
\]

\[
A + B = 0
\]

\[
A - B = 1
\]

\[
-2B = 1 \quad B = -\frac{1}{2} \quad A = \frac{1}{2}
\]

\[
\frac{\frac{1}{2}}{y-1} - \frac{\frac{1}{2}}{y+1} = \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}
\]

\[
\ln C + \frac{1}{2} \ln (y-1) - \frac{1}{2} \ln (y+1) = \frac{1}{2} \ln (x-1) - \frac{1}{2} \ln (x+1)
\]

\[
\ln c \left( \frac{y-1}{y+1} \right) = \frac{1}{2} \ln \left( \frac{x-1}{x+1} \right)
\]

\[
c \left( \frac{y-1}{y+1} \right) = \frac{x-1}{x+1}
\]

\[
y = 2 \quad x = 2
\]

\[
\ln \left[ \frac{1}{2} \right] = \frac{1}{2}
\]

\[
c = 1
\]
\[
\frac{y-1}{y+1} = \frac{x-1}{x+1}
\]

\[
x\sqrt{y-x+y-1} = x\sqrt{y-y+x-x-1} = -2x = -2y
\]

\[
x = y
\]

1 - 21 odd
23, 33
1st Order LDE's

- \( a_1(x) \frac{dy}{dx} + a_0(x) y = g(x) \)

\( g(x) = 0 \) homogeneous
\( g(x) \neq 0 \) non-homogeneous

\[ \frac{dy}{dx} \quad a_1(x) \frac{dy}{dx} + a_0(x) y = g(x) \]

forcing function system solution - response

Standard form: \( \frac{dy}{dx} + P(x) y = f(x) \)

Some important concepts. In solving DE's we often consider a non-homogeneous equation in two steps.

1. solve a corresponding homogeneous problem
2. then try to find a solution that fits the r.h. part.

If we are successful, the overall solution to the problem is:

\[ y = y_c + y_p \]

\[ \text{complementary solution} \quad \text{particular soln.} \]

\[ \text{to the given} \quad \text{to the given} \]

\[ \text{r.h. part} \quad \text{f(x).} \]
We can do this because

\[ y = y_c + y_p \]

\[
\frac{d}{dx} \left( y_c + y_p \right) + P(x) \left( y_c + y_p \right) = f(x)
\]

\[
\left[ \frac{dy_c}{dx} + P(x) y_c \right] + \frac{dy_p}{dx} + y_p \left( P(x) \right) = f(x)
\]

\[ \frac{dy_c}{dx} + P(x) y_c = 0 \]

\[
\frac{dy_c}{dx} + P(x) \, dx = 0
\]

\[
\ln y_c = -\int P(x) \, dx + c
\]

\[ y_c = c \, e^{-\int P(x) \, dx} \]

or \[ y_c = c \, y_1(x) \quad y_1(x) = e^{-\int P(x) \, dx} \]

Now introduce another concept. Use variation of parameter method.

We will assume that

\[ y_p = \boxed{U(x) \, y_1(x)} = U(x) \, e^{-\int P(x) \, dx} \]
Substitute into the standard form of a 1st order eqn.

\[ \frac{dy_p}{dx} + P(x) y_p = f(x) \]

\[ \frac{dy_e}{dx} = \left( U(x) y_1(x) + U'(x) y_1(x) \right) \]

Add intermediate step:

\[ u \left[ \frac{dy_1}{dx} + P(x) y_1 \right] + y_1 \frac{du}{dx} = f(x) \]

Since \( y_1 = ce^{\int P(x) \, dx} \)

\[ y_1 \frac{du}{dx} = f(x) \]

or

\[ du = \frac{f(x)}{y_1(x)} \, dx \]

\[ u = \int \frac{f(x)}{y_1(x)} \, dx \]

\[ y_p = uy_1 = \left[ \int \frac{f(x)}{y_1(x)} \, dx \right] e^{-\int P(x) \, dx} \]

\[ y_p = e^{-\int P(x) \, dx} \int e^{\int P(x) \, dx} f(x) \, dx \]

Combining:

\[ y = y_c + y_p \]

\[ y = c e^{-\int P(x) \, dx} + e^{-\int P(x) \, dx} \int e^{\int P(x) \, dx} f(x) \, dx \]

The solution contains both \( y_c \) and \( y_p \).
Back to our temperature sensor model:

\[ \frac{dT}{dt} + 2T = 10 \]

(already in std. form)

\[ P(t) = 2 \quad f(t) = 10 \]

\[ \int P \, dt = \int 2 \, dt = 2t \]

So, the solution is:

\[ T e^{2t} = \int 10 e^{2t} \, dt + c \]

\[ T e^{2t} = 5 e^{2t} + c \]

\[ T = 5 + ce^{-2t} \]

2.3.8

\[ \frac{dy}{dx} - 2y = \frac{x^2 + 5}{P(x)} \]

\[ SP(x) \, dx = -2x \]

\[ ye^{-2x} = \int (x^2 + 5)e^{-2x} \, dx + c \]

\[ = \int x^2 e^{-2x} \, dx + s \int e^{-2x} \, dx + c \]

\[ ye^{-2x} = -\frac{1}{2} x^2 e^{-2x} - \frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} - \frac{5}{2} e^{-2x} + c \]

\[ y = -\frac{1}{2} x^2 - \frac{1}{2} x - \frac{11}{4} + ce^{2x} \]
\[ \frac{dT}{dt} = \frac{h}{\rho} (T - T_m) \quad T(0) = T_0 \]

\[ \frac{dT}{dt} = \frac{-h}{\rho} T = \frac{-h}{\rho} T_m \quad f(t) \]

\[ SP \, dt = -h \, t \]

\[ T \, e^{-ht} = \int -h \, T_m \, e^{-ht} \, dt + C \]

\[ T \, e^{-ht} = T_m \, e^{-kt} + C \]

\[ T = T_m + C \, e^{kt} \]

\[ T(0) = T_0 \quad \Rightarrow \quad T_0 = T_m + C \]

\[ c = T_0 - T_m \]

\[ T = T_m + (T_0 - T_m) \, e^{kt} \]
given,
\[
\frac{d}{dx} (xy) = 0
\]
\[xy = c\]

but this equation could have presented itself in the following form:
\[
x \frac{dy}{dx} + y = 0 \quad \text{derivative form}
\]
or
\[
x \, dy + y \, dx = 0 \quad \text{differential form}
\]

If we could have recognized that the eyes above was the exact differential of the function \(xy\) the solution would have been easy to find, as we showed at the start.
1. **Exact Equations**

   - Suppose we have a DDE of the following form
     \[ M(x, y) + N(x, y) \frac{dy}{dx} = 0 \]

   - Now, the equation is called exact if there is some function
     \[ f(x, y) \] such that
     \[ f_x = \frac{df}{dx} = M \quad \text{and} \quad f_y = N = f_y \]

   - We can then write
     \[ \left( \frac{df}{dx} \right) + \left( \frac{df}{dy} \right) \frac{dy}{dx} = 0 \quad \text{chain rule} \]

   - We recognize that the above is the chain rule expansion of
     \[ \frac{d}{dx} \left( f(x, y) \right) \]

   - So, we get
     \[ \frac{d}{dx} \left( f(x, y) \right) = 0 \]

   - Integration yields
     \[ f(x, y) = C \]

   This would represent an implicit solution to the DDE.
Since finding this function can be somewhat difficult, it would be nice if we could a-priori determine if it even exists.

We know
\[
\frac{\partial f}{\partial x} = M \quad \frac{\partial f}{\partial y} = N
\]

Now we also know
\[
\frac{\partial f}{\partial x} \frac{\partial }{\partial y} = \frac{\partial f}{\partial y} \frac{\partial }{\partial x}
\]

So,
\[
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
\]
we can use this to test if the equation is exact.

Let's take a look at an example.

Problem 10.

\[
(x^3 + y^3) \, dx + 3xy^2 \, dy = 0
\]

Test \[
\begin{cases}
M = x^3 + y^3 \quad \frac{\partial M}{\partial y} = 3y^2 \\
N = 3xy^2 \quad \frac{\partial N}{\partial x} = 3y^2
\end{cases}
\]
equation exact

Now we need to find \( f \).

\[
\frac{\partial f}{\partial x} = M = x^3 + y^3
\]

\[
\frac{\partial f}{\partial y} = (x^3 + y^3) \, dx
\]

\[
f = \int x^3 \, dx + \int y^3 \, dx
\]
\[ f = \frac{x^4}{4} + xy^3 + h(y) \]

Now we use the definition of \( N \).

\[ N = \frac{\partial f}{\partial y} \]

So \( N = \frac{\partial}{\partial y} \left( \frac{x^4}{4} + xy^3 + h(y) \right) \)

\[ N = 3y^2x + h'(y) = 3xy^2 \]

or \( h'(y) = 0 \)

\[ h'(y) = c \]

So \( f = \frac{x^4}{4} + xy^3 + c \)

Thus, the solution is \( f(x,y) = c \)

\[ \frac{x^4}{4} + xy^3 = c \]

Problem 2.2.

\( (e^x + y) \, dx + (2 + x + ye^y) \, dy = 0 \quad y(0) = 1 \)

\[ m = e^x + y \quad N = 2 + x + ye^y \]

exact! \( \left\{ \begin{align*} \frac{\partial M}{\partial y} &= 1 \\ \frac{\partial N}{\partial x} &= 1 \end{align*} \right. \)

\[ \frac{\partial f}{\partial x} = M = e^x + y \]

\[ f = e^x + yx + h(y) \]
\[ \frac{df}{dy} = x + h'(y) = N = 2 + x + ye^y \]

\[ h'(y) = 2 + ye^y \]

\[ h(y) = 2y + ye^y - e^y \]

So,

\[ f = e^x + xy + 2y + ye^y - e^y \]

and the solution is

\[ e^x + xy + 2y + ye^y - e^y = c \]

\[ y(0) = 1 \]

\[ e^0 + 0 + 2 + e^x - e^y = c \]

\[ c = 3 \]

\[ e^x + xy + 2y + ye^y - e^y = 3 \]

**Procedure**

1. Test for exactness \( \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \)
2. Determine \( f = \int M \, dx \)
3. Find \( h'(y) \)
4. Find \( h = \int h'(y) \)
5. Compose \( f \)
6. Solve \( f = c \)