1a. Find the general solution to the following 2nd order, homogeneous differential equation.

\[ y'' + 2y' - 8y = 0 \]

\[ (m + 4)(m - 2) = 0 \]

\[ m_1 = -4 \quad m_2 = +2 \]

\[ y = C_1 e^{-4t} + C_2 e^{2t} \]

1b. Evaluate any unknown constants using the following initial conditions.

\[ y(0) = 2 \quad \text{and} \quad y'(0) = 10 \]

\[ y(0) = C_1 + C_2 = 2 \]

\[ y'(0) = -4C_1 + 2C_2 = 10 \]

\[ 4 = 6C_2 \]

\[ C_2 = 3 \]

\[ C_1 = -1 \]

1c. Show that the two parts of the general solution are linearly independent.

\[ W = \begin{vmatrix} e^{2t} & e^{-4t} \\ 2e^{2t} - 4e^{-4t} \end{vmatrix} \]

\[ = \begin{vmatrix} e^{2t} & e^{-4t} \\ 2e^{2t} - 4e^{-4t} \end{vmatrix} = -4e^{2t} - 2e^{-2t} \]

\[ = 6e^{-2t} \neq 0 \]

\[ \therefore \text{LI, or } y_1' = Ce^{-6t} + \text{constant} \]
2. The function \( y_1 \) given below is a solution to the following nonhomogeneous differential equation. Use reduction of order to find the second part of the complimentary solution, and a method of your choice to find \( y_p \). Finally, write the complete general solution.

\[
y'' + 4y = e^{2t} \quad y_1 = \cos(2t)
\]

\[
y_2 = y_1 \int \frac{-p \, dt}{y_1^2} \, dt
\]

\[
p = 0
\]

\[
y_2 = \cos(2t) \int \frac{1}{\cos^2(2t)} \, dt
\]

\[
y_2 = \cos(2t) \tan(2t) = \frac{\cos(2t) \sin(2t)}{2} = \frac{\sin(2t)}{2} \cos(2t)
\]

So,

\[
y_2 = C_2 \sin(2t)
\]

To find \( y_p \) we know that it must contain only forms of \( e^{2t} \) since then the left hand side of the e.g. would be a combination of terms containing \( e^{2t} \) and match the R.H.S.

\[
4Ae^{2t} + 4Ae^{2t} = e^{2t}
\]

\[
A = \frac{1}{8}
\]

So,

\[
y_p = \frac{1}{8} e^{2t}
\]

\[
y = C_1 \cos(2t) + C_2 \sin(2t) + \frac{1}{8} e^{2t}
\]
3.) Solve the following non-homogeneous differential equation, subject to initial conditions of \( y(0) = y'(0) = 0 \), using the method of undetermined coefficients.

\[ y'' + y' = 2x + 1 \]

\[ m^2 + m = 0 \]
\[ m(m + 1) = 0 \]
\[ m = 0 \quad m = -1 \]

\[ y_c = C_1 e^{0t} + C_2 e^{-x} = C_1 + C_2 e^{-x} \]

\[ g(x) = 2x + 1 \]
\[ g'(x) = 2 \quad g''(x) = 0 \]

\[ y_p = Ax + B \]

So \( y_p = Ax^2 + Bx \)

\[ y_p' = 2Ax + B \]
\[ y_p'' = 2A \]

\[ 2A + 2Ax + B = 2x + 1 \]

\[ \therefore \]
\[ A = 1 \]
\[ B = -1 \]

\[ y_p = x^2 - x \]

\[ y = C_2 e^{-x} + x^2 - x + C_1 \]
\[ y' = -C_2 e^{-x} + 2x - 1 \]

\[ y(0) = 0 = C_1 + C_2 \]
\[ C_1 = 1 \]
\[ C_2 = -1 \]

\[ y'(0) = 0 = -C_2 - 1 \]

\[ y = -e^{-x} + x^2 - x + 1 \]
4.) Find $y_p$ for the equation in Problem 3 using variation of parameters.

$$ W = \begin{vmatrix} 1 & e^{-x} \\ 0 & -e^{-x} \end{vmatrix} = -e^{-x} $$

$$ u_1 = -\int \frac{e^{-x} (2x+1)}{-e^{-x}} \, dx = -\int (2x+1) \, dx = -x^2 + x $$

$$ u_2 = \int \frac{2x+1}{-e^{-x}} \, dx = 2\int xe^{x} \, dx + \int e^{x} \, dx $$

$$ = -2 \left[ xe^{x} - e^{x} \right] + e^{x} = 2xe^{x} + 3e^{x} $$

$$ y_p = u_1 y_1 + u_2 y_2 $$

$$ = 1(x^2 + x) + e^{-x} \left[ 2xe^{x} + 3e^{x} \right] $$

$$ = x^2 + x - 2x + 3 $$

$$ = x^2 - x + 3 $$

matches contained in $c$.

4b.) Find the particular solution of $y'' + y' = 4x - 2$. Explain your approach.

$$ y'' + y' = -2(2x+1) $$

So multiplying $y_p$ from above by $-2$ should yield desired result since problem is linear.

$$ y_p' = -4x + 2 $$

$$ y_p'' = -4 $$

So, $-4 + (-4x + 2) = -4x - 2$.