Lecture: Section 20.3

The Curl of a Vector Field

We previously encountered the idea of the curl of a vector field back in Section 18.4. Recall that we did the curl test to determine whether a vector field was path-dependent.

In this section we will examine the concept in greater detail with an eye towards using the curl as a measure of the circulation or rotation of a vector field.

Examples of Vector Fields that can have Curl

1. **Water Flow** - In this case, a whirlpool or eddy would be a region the curl of a vector field would not equal zero.

   If the water is rotating in the xy-plane, then the direction of the curl vector would be the z-direction, i.e. the axis of rotation. Whether the direction is positive or negative is determined by the right-hand rule. In the example shown above, the rotation is counterclockwise, and the curl vector would be in the positive z-direction.

   The magnitude of the curl vector relates to the speed at which the fluid is moving around the axis of rotation.

2. **Magnetic and Electric Fields** - These fields can also have rotation. In electric fields, the curl is related to the current density.
Circulation Density

The circulation density (circulation per unit area) is defined using a closed curve. For example, consider the circle shown below with a unit normal $\vec{n}$.

The circulation density is defined as: $\text{circ}_n \vec{F} = \frac{\text{Circulation around } C}{\text{Area inside } C} = \lim_{A \to 0} \frac{\int_C \vec{F} \cdot d\vec{r}}{\text{Area}}$

The circulation density is related to the angular velocity of the rotation around $P$. This can be envisioned using a paddlewheel as shown below and on the applet by B. Terrel from Cornell (see links list on course webpage). The speed at which the paddlewheel spins is proportional to the angular velocity, and the motion is in the plane that has the normal, $\vec{n}$.

Definition of Curl

We will introduce geometric and algebraic definitions of the curl below.

Geometric:

- Direction is the direction $\vec{n}$ for which $\text{circ}_n \vec{F}$ is the greatest.
- Magnitude is the circulation density around $\vec{n}$.

Note that the first condition implies that $\text{curl} \vec{F} = \text{circ}_n \vec{F} \cdot \vec{n}$. 
Algebraic

We learned the algebraic definition in Chapter 18.

\[
\text{Curl } \vec{F} = \begin{vmatrix}
  i & j & k \\
  \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
  F_1 & F_2 & F_3
\end{vmatrix}
\]

Note \( \text{Curl } \vec{F} \) is a \( f(x,y,z) \). Therefore, it can vary with position in the field. A curl of \( \vec{0} \) would imply that the field is irrotational.

Examples

Let’s take a look at some 2D vector fields to see if we can identify whether there is a non-zero curl, and then check using the algebraic definition.

(a) \( \vec{F} = -x\hat{i} - y\hat{j} \)

This field is one that shows a negative divergence with flow into the origin. However, there appears to be no rotation.

A check of the algebraic definition confirms this:

\[
\text{Curl } \vec{F} = \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{k} = -\frac{\partial x}{\partial x} - \left( -\frac{\partial y}{\partial y} \right)
\]

\[
\text{Curl } \vec{F} = -1 + 1 = 0
\]
(b) \( \vec{F} = \cos(y) \hat{i} - x\hat{j} \)

Clearly there is no rotation in this field, which looks to be particularly strong in the region around \( x = 0 \) and \( y = -1.5 \).

Using the algebraic definition:

\[
\text{Curl } \vec{F} = \left( \frac{\partial x}{\partial x} - \frac{\partial \cos(y)}{\partial y} \right) = 1 + \sin(y)
\]

Note when \( y = \frac{\pi}{2} \), \( \text{Curl } \vec{F} = +1 \), which means that the rotation should be counterclockwise (that is, direction of axis of rotation is the positive \( z \)).

(c) \( \vec{F} = (4 - y^2) \hat{i} \)

At first glance we might think that there is zero curl in this field, however, think about what would happen to a small cylinder placed in this field (top half).

The fluid moving past the bottom of the cylinder is moving faster than at the top, which would cause a rotation.
This is borne out algebraically:

\[ \text{curl } \vec{F} = \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) = 0 - \frac{\partial (4 - y^2)}{\partial y} = +2y \hat{k} \]

So in the top half of the field, rotation would be counterclockwise (positive y) and clockwise in the bottom half.

Note that this field might be for flow of a viscous field through a pipe and the field often called a velocity profile.
Exercise 20.3.21

The figure below is a sketch of the vector field \( \vec{F} = yi + xj \). What is the direction of rotation for twigs placed on the x- and y-axes? Also, calculate the curl of the field.

\[
\begin{align*}
F &= yi + xj \\
\end{align*}
\]

(a) counterclockwise rotation of twig on x-axis

(b) clockwise rotation of twig on y-axis

(c) \( \text{Curl } \vec{F} = \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \) = \( \frac{\partial x}{\partial x} - \frac{\partial y}{\partial y} = 0 \)

As expected, there is would be no rotation of a paddlewheel.