Lecture: Section 20.1

The Divergence of a Vector Field

In this section, we study the flux through a closed surface that surrounds a point.

Flow out of a point (source). This would be like a spring.

Flow into a point (sink). This would be like a drain.
Definition of Divergence

To determine the net outflow per unit volume of a vector field at a point, we calculate the flux out of a small sphere centered at the point.

\[ \mathbf{F} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \]

We could determine the net flux through the surface of the sphere by evaluating the flux integral over the entire surface, \( S \), of the sphere.

\[ \text{Flux} = \int_{S} \mathbf{F} \cdot d\mathbf{A} \]

Now, the divergence is defined as the flux density, or flux per volume.

\[ \text{Units:} \quad \text{Flux} = \mathbf{v} \cdot \mathbf{A} = \frac{m^3}{s} \]

e.g. for velocity field:

\[ \text{Units:} \quad \text{Divergence} = \frac{\text{Flux}}{\text{volume}} = \frac{m^3}{s} = \frac{1}{s} \]
So, the physical (geometric) definition of divergence is (evaluated as V goes to infinity):

\[
div \vec{F} = \lim_{V \to \infty} \frac{\int_{S} \vec{F} \cdot d\vec{A}}{V}
\]

**Example 1:** Page 940

Determine the divergence of a vector field \( \vec{F} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \)

Recall that we already solved this problem in Section 19.2, and saw:

\[
\int_{S} \vec{F} \cdot d\vec{A} = \vec{F}(R, \theta, \phi) \cdot [\sin\phi \cos\theta \hat{i} + \sin\phi \sin\theta \hat{j} + \cos\phi \hat{k}] R^2 \sin\phi \, d\phi d\theta
\]

Which simplified to:

\[
\text{Flux} = 4\pi a^3
\]

To find the divergence of \( \vec{F} \):

\[
div \vec{F} = \lim_{a \to 0} \frac{\int_{S} \vec{F} \cdot d\vec{A}}{V}
\]

\[
= \lim_{a \to 0} \frac{4\pi a^3}{3\pi a^3} = \lim_{a \to 0} \frac{4}{3} = \frac{4}{3}
\]

So the divergence is a positive number for this case as we may have expected.

**Algebraic Definition of Divergence**

The algebraic definition of divergence is:

\[
div \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}
\]

Note that this form makes some intuitive sense. Each term above \( \frac{\partial F_i}{\partial x_i} \) is telling us how the field is changing in a given direction. Note that for there to be a non zero divergence in a given direction, the field must be a function of the direction. This makes sense if there is to be a net in/out flow at a point. For example consider the vector field

\[\vec{F} = x^2 \hat{i} + y^2 \hat{j}\]
In this case, \( \text{div} \mathbf{F} = 2x + 2y \). The divergence is shown as contours imposed on the field plot below, which is an interesting way of considering this scalar quantity. Note that the divergence, in this case, has the form of a plane. Therefore, the contours are parallel lines. Observe that each of these contours would be lines of constant inflow/outflow. It is seen that in the lower left corner, the contours are negative, which means that there would be a net inflow, whereas in the upper right portion of the graph the positive contours suggest net outflow. This is consistent with what would be concluded from the appearance of the field vectors.

Now, let’s check units. If \( \mathbf{F} \) is a velocity field, for example, \( \frac{\partial F_i}{\partial x_i} \) would have units of \( \frac{m}{s} \).

So, \( \text{div} \mathbf{F} \) would have units of \( \frac{1}{s} \) as expected.

**Example 2:** Calculate the divergence of the following vector fields

(a) \( \mathbf{F} = x \hat{i} + y \hat{j} \)

(b) \( \mathbf{F} = \cos x \hat{i} + \sin y \hat{j} \)

(c) \( \mathbf{F} = y \hat{i} + x \hat{j} \)
Ans:

(a) \( \text{div} \vec{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 2 \)

(b) \( \text{div} \vec{F} = \frac{\partial \cos x}{\partial x} + \frac{\partial \sin y}{\partial y} = -\sin x + \cos x \)

(c) \( \text{div} \vec{F} = \frac{\partial y}{\partial x} + \frac{\partial x}{\partial y} = 0 \)