Lecture: Section 19.2

Flux Integrals for Graphs, Cylinders, and Spheres

In this section, we will learn how to compute flux through surfaces that are graphs of functions, cylinders, and spheres.

Flux through $z = f(x, y)$

\[ \text{Area vector of parallelogram} = \vec{A} = \vec{v}_x \times \vec{v}_y \]
Consider the patch of surface above a region \( \Delta x \) by \( \Delta y \) in the \( xy \)-plane.

Recall that the area of a parallelogram can be determined by the cross product of the vectors that form the sides.

So approximating the area of the surface as shown in Figure 19.20, we have

\[
A \approx \vec{v}_x \times \vec{v}_y
\]

We can find vectors in directions of \( \vec{v}_x \) and \( \vec{v}_y \) by taking the partials of the position vector \( \vec{r} = x\hat{i} + y\hat{j} + f(x, y)\hat{k} \) with respect to \( x \) and \( y \).

\[
\vec{r}_x = \frac{\partial \vec{r}}{\partial x} = \hat{i} + f_x\hat{k}
\]

\[
\vec{r}_y = \frac{\partial \vec{r}}{\partial y} = \hat{j} + f_y\hat{k}
\]

Now, \( \vec{r}_x \) has a component of \( \hat{i} \) in the \( x \)-direction, whereas we know that \( \vec{v}_x \) has a component of \( \Delta x\hat{i} \).

Thus,

\[
\vec{r}_x \Delta x = \vec{v}_x
\]

And, similarly for \( \vec{v}_y \)

\[
\vec{v}_y = \vec{r}_y \Delta y
\]

Finally,

\[
\Delta A = \vec{v}_x \times \vec{v}_y = (\vec{r}_x \times \vec{r}_y) \Delta x \Delta y = (-f_x\hat{i} - f_y\hat{j} + \hat{k}) \Delta x \Delta y
\]

In the limit as \( \Delta x \) and \( \Delta y \to 0 \)

\[
dA = (-f_x\hat{i} - f_y\hat{j} + \hat{k}) \, dx \, dy
\]

And the flux integral is:

\[
\int_S \vec{F} \cdot d\vec{A} = \int S \vec{F} (x, y, f(x, y)) \cdot (-f_x\hat{i} - f_y\hat{j} + \hat{k}) \, dx \, dy
\]
Example: Exercise 19.2.4

Calculate the flux of the vector field \( \vec{F} = 2x\hat{j} + y\hat{k} \) through \( z = -y + 1 \) above the square \( 0 \leq x \leq 1 \) and \( 0 \leq y \leq 1 \) oriented upward.

\[
\vec{F} = -y + 1 \quad f_x = 0 \quad f_y = -1
\]

\[
\int_S \vec{F} \cdot d\vec{A} = \int_S (2x\hat{j} + y\hat{k}) \cdot (\hat{j} + \hat{k}) \, dx \, dy = \int_S 2x + y
\]

\[
= \int_0^1 \int_0^1 (2x + y) \, dx \, dy = \int_0^1 (x^2 + xy) \bigg|_0^1 \, dy = \int_0^1 (1 + y) \, dy = \frac{y^2}{2} + y \bigg|_0^1 = \frac{3}{2}
\]

The Flux of a Vector Field Through a Cylinder

\[
\Delta A \approx R\Delta \theta \Delta z
\]

The outward unit normal \( \vec{n} \) points in the direction of \( x\hat{i} + y\hat{j} \), so

\[
\vec{n} = \frac{x\hat{i} + y\hat{j}}{\|x\hat{i} + y\hat{j}\|} = \frac{R \cos(\theta) \hat{i} + R \sin(\theta) \hat{j}}{R} = \cos(\theta)\hat{i} + \sin(\theta)\hat{j}
\]
A similar analysis can be done for a cylinder, and it is reasonably straightforward to show \( d\vec{A} = (\cos(\theta) \hat{i} + \sin(\theta) \hat{j}) \, R \, dzd\theta \) where \( R \) is the radius of a cylinder.

So to compute the flux through the sidewalls of a cylinder, we have:

\[
\int_S \vec{F} \cdot d\vec{A} = \int_T \vec{F}(R, \theta, z) \cdot (\cos(\theta) \hat{i} + \sin(\theta) \hat{j}) \, R \, dzd\theta
\]

Where \( T \) is the \( \theta \, z \) region corresponding to \( S \).

Example: Exercise 19.2.14

Compute the flux through a cylindrical surface with \( R = 1 \) that is 6 units long, due to the vector field \( \vec{F} = xz\hat{i} + yz\hat{j} + z^3\hat{k} \), oriented away from the \( z \)-axis.

\[
\int_S \vec{F} \cdot d\vec{A} = \int_T (xz\hat{i} + yz\hat{j} + z^3\hat{k}) \cdot (\cos(\theta) \hat{i} + \sin(\theta) \hat{j}) \, (1) \, dzd\theta
\]

Next, replace:

\[ x = R \, \cos(\theta) = \cos(\theta) \, , \, y = R \, \sin(\theta) = \sin(\theta) \]
\[
= \int_T (\cos^2 \theta \cdot z + \sin^2 \theta \cdot z) \cdot dzd\theta
\]

Note, \( z^3 \) disappears since there is no \( \hat{k} \) component in \( d\mathbf{A} \).

\[
= \int_0^{2\pi} \int_0^6 z \, dz \, d\theta = \int_0^{2\pi} \left[ \left( \frac{z^2}{2} \right) \right]_0^6 \, d\theta = \int_0^{2\pi} 18 \, d\theta = 36\pi
\]

Flux Through a Sphere

The flux through a sphere is:

\[
\int_S \mathbf{F} \cdot d\mathbf{A} = \int_T \mathbf{F}(R, \theta, \phi) \cdot (\sin(\phi) \cos(\theta)\hat{i} + \sin(\phi) \sin(\theta)\hat{j} + \cos(\phi)\hat{k}) R^2 \sin(\phi) \, d\phi d\theta
\]

Example

Compute the flux due to \( \mathbf{F} = x\hat{i} + y\hat{j} + z\hat{k} \) through the sphere \( x^2 + y^2 + z^2 = a^2 \).

\[
\int_S \mathbf{F} \cdot d\mathbf{A} = \int_T (a \sin(\phi) \cos(\theta)\hat{i} + a \sin(\phi) \sin(\theta)\hat{j} + a \cos(\phi)\hat{k}) \cdot (\sin(\phi) \cos(\theta)\hat{i}
+ \sin(\phi) \sin(\theta)\hat{j} + \cos(\phi)\hat{k}) a^2 \sin(\phi) \, d\phi d\theta
\]

\[
= \int_T (a \sin^2(\phi) \cos^2(\theta) + a \sin^2(\phi) \sin^2(\theta) + a \cos^2(\phi)) a^2 \sin(\phi) \, d\phi d\theta
\]

\[
= \int_T a [a \sin^2(\phi) \cos^2(\theta) + a \sin^2(\phi) \sin^2(\theta) + a \cos^2(\phi) a^2 \sin(\phi)] \, d\phi d\theta
\]

\[
= \int_T a [\sin^2(\phi) + \cos^2(\phi)] a^2 \sin(\phi) \, d\phi d\theta
\]

\[
= \int_T a^3 \sin(\phi) \, d\phi d\theta = a^3 \int_0^{2\pi} \int_0^\pi \sin(\phi) \, d\phi d\theta = a^3 \int_0^{2\pi} (-\cos (\phi) \bigg|_0^\pi) d\theta
\]

\[
= a^3 \int_0^{2\pi} 2 \, d\theta = \frac{4\pi a^3}{3}
\]