Lecture: Section 19.1

Flux (Surface) Integrals

In chapter 18, we learned how to integrate vector fields along curves (line integrals).

In chapter 19, we will integrate a vector field over a surface. If the vector field represents a flowing fluid, this integration would yield the rate of flow through the surface, or flux.

We can also compute the flux of an electric or magnetic field. Even though no flow is taking place, the concept is the same.

Orientation of Surface and Area Vector

Consider the flow through the two areas shown below.
The area vector of a flat, oriented surface is a vector \( \mathbf{A} \) such that

- The magnitude of \( \mathbf{A} \) is the area of the surface
- The direction of \( \mathbf{A} \) is in the direction of the orientation unit normal vector, \( \mathbf{n} \).

So, \( \mathbf{A} = A\mathbf{n} \), which means that the surface has been assigned an orientation. This will be helpful for a couple of reasons. First, it will allow us to easily determine the area that is open to flow. Clearly, in the figure above, the upper shape has a greater fraction of the total area open to flow than the lower one. We can also infer that, if the flow and the area normal are at right angles, then the flux will be zero, since there is no area available. Also, the flux is a scalar quantity, so we need some way to determine the sign. Specifying an orientation for the area allows us to do this. If the flow and the area normal are aligned, then the flux will be positive.

**Flux of a Constant Vector Field Through a Flat Surface**

![Diagram of a flat surface with a velocity vector and area vector]

The area available for flow through the box is the component of the area in the direction of the flow, i.e., \( A\cos(\theta) \)

So the flux is:

\[
Flux = ||\mathbf{v}|| ||\mathbf{A}|| \cos(\theta) = \mathbf{v} \cdot \mathbf{A}
\]

First consider the units. If the velocity is in units of length/time and area has units of length squared, then the flux will have units of (length\(^3\)/time), which is a volumetric flow rate.

Second, we must understand that this equation is quite limited in nature, i.e., to constant velocity flows through flat surfaces. In order to consider more complicated situations, we will need to integrate the flow over the surface.
Example: Exercise 19.1.8

Calculate the flux of the vector field \( \vec{F} = -5\hat{i} \) through a square of side length 3, centered on the x-axis at \( x = -5 \). The area oriented in the positive x direction.

\[
\vec{F} = -5\hat{i} \quad \vec{A} = 9\hat{i}
\]

\[
Flux = \vec{F} \cdot \vec{A} = -5\hat{i} \cdot 9\hat{i} = -45
\]

The Flux Integral

We will now consider the flux of a vector field \( \vec{F} \) which may not be constant through a curved surface.

We start by dividing the surface into small patches.
So we can write the oriented area vector $\Delta \vec{A}$ as $\Delta \vec{A}_i$ for each patch. Now if we make the patches small enough, we can approximate the flux through the patch as

$$Flux_i = \sum \vec{F}_i \cdot \Delta \vec{A}_i$$

*in the limit as $\Delta \vec{A}_i \to 0$

$$\sum \vec{F}_i \cdot \Delta \vec{A}_i = \int_S \vec{F} \cdot d\vec{A}$$

The $\int_S \vec{F} \cdot d\vec{A}$ is called a flux integral in our text. It is also sometimes referred to as a surface integral, for obvious reasons.

**Example: Exercise 19.1.18**

Calculate the flux through the rectangle $x = 4$, $-1 \leq y \leq 1$, $-1.5 \leq z \leq 1.5$, oriented in the positive x-direction, and $\vec{F} = (x + 3)\hat{i} + (y + 5)\hat{j} + (z + 7)\hat{k}$.

$$Flux = \int_S \vec{F} \cdot d\vec{A} \quad d\vec{A} = \hat{i}dA$$
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\[ Flux = \int_S [(x + 3)i + (y + 5)j + (z + 7)\mathbf{k}] \cdot d\mathbf{A} \]

But we are told \( x = c = 4 \)

\[ Flux = \int_S (x + 3) d\mathbf{A} \]

Note, in reality this is a double integral:

\[ 7 \int_S d\mathbf{A} = 7 \int_{-1.5}^{1.5} \int_{-1}^{1} dydz = 7 \int_{-1.5}^{1.5} (y \bigg|_{-1}^{1}) dz = 7 \cdot 2 \int_{-1.5}^{1.5} z dz = 7 \cdot 2 \cdot 3 = 42 \]

Example: Exercise 19.1.24
Find the flux if \( \mathbf{F} = \cos (x^2 + y^2) \mathbf{k} \) and the area is the disk \( x^2 + y^2 \leq 9 \) oriented upward in the plane \( z = 1 \).
Flux = \int_S \cos(x^2 + y^2) \mathbf{\hat{k}} \cdot \mathbf{\hat{k}} \, dA = \int_S \cos(x^2 + y^2) \, dA

Now use polar coordinates: \( r^2 = x^2 + y^2 \), \( dA = r \, dr \, d\theta \)

\[
\int_S \cos(x^2 + y^2) \, dA = \int_S (\cos r^2) \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^3 \cos(r^2) \, r \, dr \, d\theta
\]

\[
\int_0^3 \cos(r^2) \, r \, dr = u = r^2, \ du = 2r \, dr
\]

\[
= 0.5 \sin(r^2) \bigg|_0^3 = 0.5[\sin(9) - \sin(0)] = \frac{\sin(9)}{2}
\]

\[
\int_0^{2\pi} \frac{\sin(9)}{2} \, d\theta = \left. \frac{\sin(9)}{2} \theta \right|_0^{2\pi} = \frac{\pi \sin(9)}{2}
\]