Lecture: Section 18.3

Gradient Field and Path Independent Fields

For this section we will follow the treatment by Professor Paul Dawkins (see http://tutorial.math.lamar.edu) more closely than our text. A link to this tutorial is on the course webpage.

Recall from Calc I, the Fundamental Theorem of Calculus.

\[ \int_{a}^{b} F'(x)\,dx = F(b) - F(a) \]

There is, in fact, a Fundamental Theorem for line Integrals over certain kinds of vector fields.

Note that in the case above, the integrand contains \( F'(x) \), which is a derivative. The multivariable analogy is the gradient. We state below the Fundamental Theorem for line integrals without proof.

\[ \int_{c} \vec{V} f \cdot d\vec{r} = f(Q) - f(P) \]

Where \( P \) and \( Q \) are the endpoints of the path \( c \).

A logical question to ask at this point is what does this have to do with the material we studied in sections 18.1 and 18.2?

- First, remember that \( \vec{V} f \) is a vector field, i.e., the direction/magnitude of the maximum change of \( f \).

- So, if we had a vector field \( \vec{F} \) which we know was a gradient field of some function \( f \), we would have an easy way of finding the line integral, i.e., simply evaluate the function \( f \) at the endpoints of the path \( c \).

\[ \rightarrow \text{The problem is that not all vector fields are gradient fields.} \]

So we will need two skills in order to effectively use the Fundamental Theorem for line integrals fully.
1. Know how to determine whether a vector field is a gradient field. If it is, it is called a \textit{conservative} vector field. For such a field, there is a function \( f \) such that 
\[ \vec{F} = \nabla f. \]

2. If we determine that \( \vec{F} \) is a conservative vector field, how do we find \( f \)? Note, \( f \) is called a \textit{potential} function.

\textbf{Ramifications of a conservative field:}

If \( \vec{F} = \nabla f \),

1. Then \( \int_c \vec{F} \cdot d\vec{r} = \int_c \nabla f \cdot d\vec{r} = f(Q) - f(P) \)

\begin{align*}
\int_{c_1} \vec{F} \cdot d\vec{r} &= \int_{c_2} \vec{F} \cdot d\vec{r} \\
\Rightarrow \int_{c_1} \vec{F} \cdot d\vec{r} &= \int_{c_2} \vec{F} \cdot d\vec{r}
\end{align*}

This implies:

\[ \int_{c_1} \vec{F} \cdot d\vec{r} = \int_{c_2} \vec{F} \cdot d\vec{r} \]

Which means that the line integral is path-independent.

2. Another important ramification occurs when the path \( c \) is closed. Then,

\[ \oint_c \vec{F} \cdot d\vec{r} = \oint_c \nabla f \cdot d\vec{r} = f(Q) - f(P) \]
But if \( P = Q \), then:

\[
\oint_C \nabla f \cdot d\mathbf{r} = f(P) - f(P) = 0
\]

Test for a Conservative Vector Field

Theorem: Let \( \vec{F} = F_1 \hat{i} + F_2 \hat{j} \) be a vector field on an open region \( R \). Then, if \( F_1 \) and \( F_2 \) have continuous first order partial derivatives such that

\[
\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}
\]

(Test for a conservative vector field)

then \( \vec{F} \) is a conservative vector field.

The basis for this test is easy to show. We know that if we have a \( \vec{F} \) is a gradient vector field then \( F_1 \) and \( F_2 \) in the equation above are equal to \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \), respectively, where \( f \) is the potential function. Accordingly,

\[
\frac{\partial F_1}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} \quad \text{and} \quad \frac{\partial F_2}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}
\]

But, we know that the two mixed second derivatives are equal to each other, and the basis for the test is clear.

Example: Exercise 18.3.4

Decide whether the following vector field could be a gradient field (i.e., is it conservative?).

\[
\vec{F}(x, y) = (x^2 - y^2)\hat{i} - 2xy\hat{j}
\]

\[
F_1 = x^2 - y^2 \quad F_2 = -2xy
\]

\[
\frac{\partial F_1}{\partial y} = -2y = \frac{\partial F_2}{\partial x} = -2y
\]

So we would conclude that \( G(x, y) \) is, in fact, conservative.
How to Determine $f$ for a Conservative Vector Field, $\vec{F}$

For the case above, we know now that:

$$\vec{F}(x, y) = \vec{\nabla}f = (x^2 - y^2)\hat{i} - 2xy\hat{j}$$

The question now is to find $f$.

Well, we know from the definition of the gradient,

$$\vec{\nabla}f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j}$$

So then,

$$\frac{\partial f}{\partial x} = F_1 = x^2 - y^2 \quad \text{and} \quad \frac{\partial f}{\partial y} = F_2 = -2xy$$

We could integrate these functions, in a special way, to find $f$.

$$f = \int (x^2 - y^2)\,dx = \int (-2xy)\,dy$$

Let’s take the first case:

$$\int (x^2 - y^2)\,dx = \frac{x^3}{3} - y^2x + h(y) = f$$

Note we treat $y$ as a constant, and as such, our constant of integration may, in fact, be a function of $y$.

Now, if we differentiate this expression with respect to $y$, see what happens.

$$\frac{\partial f}{\partial y} = -2yx + h’y$$

Which we know must equal $F_2$ since $\vec{F}$ is a gradient field. Therefore:

$$-2xy + h’y = F_2 = -2xy$$

So $h’(y) = 0$ and $h(y) = c$. Thus $f = \frac{x^3}{3} - y^2x + c$. Note that since we calculate $f(x,y)$ to be used in The Fundamental Theorem of Line Integrals, the $c$ value will cancel in subsequent calculations. This will be shown in the following example.
Application

Find $\int_c \vec{F} \cdot d\vec{r}$ along a parabolic path from (0, 0) to (2, 4). Well we know that the field is conservative and path independent (so no need to parameterize). Moreover, we know; 

if. $\vec{F}(x,y) = (x^2 - y^2)\hat{i} - 2xy\hat{j}$ then $f = \frac{x^3}{3} - y^2x + c$  ; So, 

$$\int_c \vec{F} \cdot d\vec{r} = f(2,4) - f(0,0) = \left[\frac{8}{3} - 32 + c\right] - [c] = \frac{-88}{3}$$

Three Dimensional Case

We don’t have a method yet to determine whether a 3-D vector field is conservative. We’ll get that in Section 18.4. But if we know we have a gradient field, we can find f.

Example: Exercise 18.3.12

Find $\int_c \vec{F} \cdot d\vec{r}$, $\vec{F} = 2x\hat{i} - 4y\hat{j} + (2z - 3)\hat{k}$, with C between (1, 1, 1) and (2, 3, -1)

$$\frac{\partial f}{\partial x} = 2x \quad \frac{\partial f}{\partial y} = -4y \quad \frac{\partial f}{\partial z} = (2z - 3)$$

We are lucky in that each of the partials are functions of a single variable.

$$f = \int 2x \, dx = x^2 + g(y,z)$$

$$\frac{\partial f}{\partial y} = g_y(y,z) = -4y$$

$$g_y(y,z) = \int -4y \, dy = -2y^2 + h(z)$$

$$F = x^2 - 2y^2 + h(z)$$

$$\frac{\partial f}{\partial z} = h'(z) = 2z - 3$$

$$h = z^2 - 3z + c$$

$$f = x^2 - 2y^2 + z^2 - 3z + c$$

$$\int_c \vec{F} \cdot d\vec{r} = f(2,3,-1) - f(1,1,1) = 4 - 18 + 1 + 3 + c - (1 - 2 + 1 - 3 + c) = -10 - (-3) = 7$$