Lecture: Section 16.5

Cylindrical and Spherical Coordinates

These systems provide alternative to Cartesian coordinates to describe volumes. We will see that cylindrical coordinates have common aspects with the 2D polar coordinates. Spherical coordinates are new to us.

Cylindrical Coordinates

We will use $r$ and $\theta$ to define a point, line, or area in the x-y plane. Next we consider a polar area element $(rdrd\theta)$, that we will slide along the z-axis. This process will form a volume.

$$V = \int_{w}^{r} rdrd\theta dz$$

So we can easily write the pertinent relationships between Cartesian and cylindrical coordinates.

$$
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta \\
r^2 &= x^2 + y^2
\end{align*}
$$

$$z = z$$
Lecture 16.5

It is useful to visualize cylindrical coordinates by sketching surfaces obtained by setting one variable to a constant and letting the others vary.

\[ r = c \]

\[ \theta = c \]

\[ z = c \]

These are the fundamental surfaces for cylindrical coordinates.

Example: Exercise 16.5.7
Find the limits for \( r, \theta, \) and \( z \) to describe the volume of the region below.

\[ 0 \leq r \leq 4 \]
\[ 0 \leq \theta \leq 2\pi \]
\[ 0 \leq z \leq 1 \]
Now find the volume of the region.

\[ V = \int_R dV = \int_0^1 \int_0^{2\pi} \int_0^4 r \, dr \, d\theta \, dz \]

\[ \int_0^4 r \, dr = \frac{r^2}{2} \bigg|_0^4 = 8 \]

\[ \int_0^{2\pi} 8 \, d\theta = 8 \theta \bigg|_0^{2\pi} = 16\pi \]

\[ \int_0^1 16\pi \, dz = 16\pi z \bigg|_0^1 = 16\pi \]

Check using geometry: \[ V = \pi r^2 h = \pi (4^2)(1) = 16\pi \]

**Example**

Integrate \( f(x, y) = \sin(x^2 + y^2) \) over the cylindrical region previously discussed.

We see that this would be tough to do in Cartesian coordinates due to the \( x^2 \) and \( y^2 \) in the argument of the sine.

We recall that \( x^2 + y^2 = r^2 \), so \( f(x, y) = f(r) = \sin(r^2) \)

We already know the limits of integration.

\[ \int_W f(x, y) \, dxdydz = \int_W f(r) \, rdrd\theta dz \]

\[ = \int_0^1 \int_0^{2\pi} \int_0^4 \sin(r^2) \, dr \, d\theta \, dz \]

\[ \int_0^4 \sin(r^2) \, r \, dr = -\frac{1}{2} \cos(r^2) \bigg|_0^4 = \frac{1}{2} [1 - \cos(4)] \]

\[ \int_0^{2\pi} \frac{[1 - \cos(4)]}{2} \, d\theta = \pi (1 - \cos(4)) \]

\[ \int_0^1 \pi (1 - \cos(4)) \, dz = [\pi(1 - \cos(4))] \]
Spherical Coordinates

In this case we will use the length \( \rho \) (the radius of a sphere) and two angels to describe the position of a point.

\[
\begin{align*}
\rho &= \text{length} \\
\theta &= \text{angle with } x\text{-axis} \quad 0 \leq \theta \leq 2\pi \\
\phi &= \text{angle with } z\text{-axis} \quad 0 \leq \phi \leq \pi
\end{align*}
\]

Note the following relationships:

\[
\begin{align*}
sin\phi &= \frac{r}{\rho} & r &= \rho \sin\phi \\
x &= \rho \sin\phi\cos\theta & y &= \rho \sin\phi\sin\theta \\
\cos\phi &= \frac{z}{\rho} & \rho &= z \cos\phi
\end{align*}
\]

From the Pythagorean Theorem:

\[
\rho^2 = r^2 + z^2 = x^2 + y^2 + z^2
\]
Fundamental Surfaces of Spherical Coordinates

\[ \rho = c \]
\[ \theta = c \]
\[ \phi = c \]

Volume Element in Spherical Coordinates

\[ \rho \sin(\phi) \Delta \rho \Delta \phi \Delta \theta \]
Again, we’ll make use of the arc length, which is the radius times the angle.

So the differential volume element is obtained by multiplying the sides of the region we will consider a cube.

\[
\Delta V = (\rho \sin \phi \Delta \theta)(\rho \Delta \phi)(\Delta \rho) = \rho^2 \sin \phi \Delta \rho \Delta \theta \Delta \phi
\]

In the limit as \(\Delta \rho, \Delta \phi,\) and \(\Delta \theta\) approach zero:

\[
dV = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi
\]

Example: Exercise 16.5.20

Cone topped by a sphere. Radius = 1 and centered at origin. Set up the volume integral in Cartesian, cylindrical, and spherical coordinates.
(a). Cartesian

The equation for the zone is \( z = \sqrt{x^2 + y^2} \) and the sphere is \( x^2 + y^2 + z^2 = 1 \)

Find the intersection of the cone and sphere, where \( z^2 = z^2 \)

\[
x^2 + y^2 = 1 - (x^2 + y^2)
\]
\[
2(x^2 + y^2) = 1
\]
\[
x^2 + y^2 = \frac{1}{2}
\]

So the intersection is a circle of radius \( \frac{1}{\sqrt{2}} \)

\[
y = \frac{1}{\sqrt{2}} - x^2
\]

Now we can set up the integral.

\[
\int \int \int_V \, dV = \int_{1/\sqrt{2}}^{1/\sqrt{2}} \int_{1/2-x^2}^{1/2-x^2} \int_{\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \, dz \, dy \, dx
\]

(b). Cylindrical

The equation of the cone is \( z = r \)

The equation of the sphere \( r^2 + z^2 = 1 \) \( \to z = \sqrt{1-r^2} \)

(Note: \( r^2 = x^2 + y^2 \))

We already found the radius of the cone in the last problem. We could do it again by setting \( z^2 = z^2 \).

\[
r^2 = 1 - r^2
\]
\[
2r^2 = 1 \to r = \frac{1}{\sqrt{2}}
\]

Therefore,

\[
V = \int_0^{2\pi} \int_0^{1/\sqrt{2}} \int_r^\sqrt{1-r^2} \, r \, dz \, dr \, d\theta
\]
(c). Spherical

Equation for the sphere is $\rho = 1$

We see that $\phi = \frac{\pi}{4}$ and $\theta = 2\pi$

Therefore,

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

MUCH EASIER!

Set up the integral of $f(x,y) = 1$ over the volume inside the paraboloid $z = 25 - x^2 - y^2$ and outside the cylinder $x^2 + y^2 = 9$ above the xy plane. First we will take a look at a figure of these surfaces.

The volume of interest will be of a “collar” shape, that will be bounded by the xy plane on the bottom and the paraboloid on the top. In order to better visualize the area over which we will integrate, the figure above was rotated in order to view it from the top. The result is shown below.
It can be seen, that the base is the annular region between circles of radius 3 and radius 5. Due to the circular nature of the base, it is easiest to set this problem up in terms of cylindrical coordinates.

We see that the z variable ranges between the xy plane \((z = 0)\) and the paraboloid \((z = 25 - x^2 - y^2 = 25 - r^2)\). The \(r\) variable ranges between the inside circle \((r = 3)\) and the outside circle \((r = 5)\). Finally, since we are integrating over the entire circle, the limits on theta are \(0\) and \(2\pi\). So we can write the integral as;

\[
V = \int_0^{2\pi} \int_3^5 \int_0^{25-r^2} r \, dz \, dr \, d\theta
\]