Lecture: Section 16.4

Double Integrals in Polar Coordinates

It is easier to work in polar coordinates when dealing with circular areas. As we have seen, trying to integrate over circular areas using rectangular coordinates leads to some messy limits containing terms such as \( \sqrt{r^2 - x^2} \).

Polar coordinates will help us to simplify matters. Recall that it is possible to define a position on a Cartesian plane with a distance and an angle.

\[
\begin{align*}
  r &= \text{distance to point} & \theta &= \text{angle from positive } x\text{-axis}
\end{align*}
\]

We can then use trigonometry to define relationships between Cartesian (rectangular) and polar coordinates.

\[
\begin{align*}
  \cos \theta &= \frac{x}{r} & r \cos \theta &= x \\
  \sin \theta &= \frac{y}{r} & r \sin \theta &= y
\end{align*}
\]

Also, using the Pythagorean Theorem:

\[
x^2 + y^2 = r^2
\]
Formation of $dA$ in Polar Coordinates

Consider the area element shown below that is produced by sweeping a line of length $\Delta r$ through an angle of $\Delta \theta$.

Recall that an arc length can be calculated by multiplying an angle in radians by the arc length. If $\Delta r$ and $\Delta \theta$ are both small, the area of $\Delta A$ is approximately that of a rectangle with sides $\Delta r$ and $r \Delta \theta$.

So $\Delta A = r \Delta r \Delta \theta$

In the limit as $\Delta r$ and $\Delta \theta$ approach zero,

$$dA = r \, dr \, d\theta$$
Example: Find the area between the semicircles shown below

We see that $r$ ranges between 1 and 2 and $\theta$ between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$

$$A = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{1}^{2} r \, dr \, d\theta$$

$$\int_{1}^{2} r \, dr = \frac{r^2}{2} \bigg|_{1}^{2} = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{3\pi}{2} d\theta = \frac{3}{2} \left[ \frac{3\pi}{2} - \frac{\pi}{2} \right] = \frac{3\pi}{2}$$

This makes sense since the area between full circles of radii 2 and 1 is

$$\pi r_2^2 - \pi r_1^2 = 4\pi - \pi = 3\pi$$

For the situation above, we have one half of this area, so $A = \frac{3\pi}{2}$. 
Example: Sketch the region over which the integral below is computed.

\[
\int_{\pi/4}^{\pi/2} \int_{0}^{2/sin\theta} f(r, \theta) \, r \, dr \, d\theta
\]

We see that \( r \) ranges between zero and \( \frac{2}{sin\theta} \)

\[
r = \frac{2}{sin\theta} \quad 2 = rsin\theta = y
\]

Also, \( \theta \) ranges between \( \frac{\pi}{4} \) and \( \frac{\pi}{2} \).

Note: this exercise is for demonstration purposes only. Just as one would likely not generally specify a circular problem using rectangular coordinates, it would not be best practice to specify use polar coordinates for the problem above, which would be better handled using rectangular coordinates.
Evaluation of a Double Integral in Polar Coordinates

\[ \int_R \sin(x^2 + y^2) \, dA \]

First of all, we see that \( R \) is a disk of radius 2, centered at the origin. The geometry of the problem should suggest that we consider using polar coordinates. If we choose to go in this direction we would need to convert the integrand from \( f(x,y) \) to \( g(r, \theta) \). This is easy to do in the present case since \( x^2 + y^2 = r^2 \).

So we make this substitution into the integral and set the limits \( 0 \leq r \leq 2 \) for the radius and \( 0 \leq \theta \leq 2\pi \) for the angle. Finally, we insert the polar area element, \( r \, dr \, d\theta \), and then we are ready to evaluate the integral.

\[
\int_0^{2\pi} \int_0^2 (\sin r^2) r \, dr \, d\theta
\]

\[
\int_0^2 (\sin r^2) \, r \, d\theta \quad u = 2r \quad du = 2r \, dr
\]

\[
= \frac{1}{2} \int \sin(u) \, du = -\frac{1}{2} \cos r^2 \bigg|_0^2 = \frac{1 - \cos 4}{2}
\]

\[
\frac{1}{2} \int_0^{2\pi} (1 - \cos(4)) \, d\theta = \frac{2\pi}{2} (1 - \cos(4)) = \pi (1 - \cos(4))
\]