The Definite Integral of a Function of Two Variables

Recall integration of single variable functions.

We defined the definite integral:

\[ \int_a^b f(x) \, dx = \lim_{\Delta x \to 0} \sum_i f(x_i) \Delta x \]

We estimated the definite integral at finite values of \( \Delta x \).

\[ \int_a^b f(x) \, dx \approx \sum_i f(x_i) \Delta x \]

Effectively we were summing rectangles of width \( \Delta x \) and height \( f(x_i) \). The integral represents the area beneath the curve \( f(x) \) and the x-axis between the limits \( x = a \) and \( x = b \).
We can extend this concept to a function of two variables.

Volume under a Surface

In this case, however, instead of finding the area of rectangular strips, we will find the volume by summing volumes of rectangular solids (prisms) with base dimensions $\Delta x$ and $\Delta y$. The height of these prisms will be $f(x, y)$.

Now, if $\Delta x$ and $\Delta y$ are finite, and the surface is not parallel to the base, the $f(x, y)$ will vary over the base.
So we will have the same issue as we did for single variable functions, i.e., which $f(x, y)$ value to choose. Since we will estimate the volume as a rectangular prism (top and bottom faces parallel), this will result in an over-estimate or under-estimate of the actual volume.

Let’s take a look at an example.

**Exercise 16.1.25**

The table below gives values of $f(x, y)$, the number of mosquito larvae per square meter, in a swamp. The dimensions $x$ and $y$ are in meters, and the region of interest $R$ is $0 \leq x \leq 8$ and $0 \leq y \leq 6$.

Estimate $\int_R f(x, y)\,dA$.

<table>
<thead>
<tr>
<th>y</th>
<th>0</th>
<th>3</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>5</td>
<td>9</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>9</td>
<td>15</td>
</tr>
</tbody>
</table>

The numbers in the table essentially represent heights above a base area of 6 meters ($y$) by 8 meters ($x$).

The easiest way to visualize this is to look at the situation from above.
The figure below shows the rectangular prisms that would result by basing the volume estimate for Quadrant I on the low and high $f(x,y)$ values, respectively.
In this case, $\Delta x = 4$ meters and $\Delta y = 3$ meters.

We will next write the height of each corner of the columns above the $x$-$y$ plane on the diagram.

The volume in a rectangular prismatic column in this example will be:

$$V_i = \Delta x \Delta y f(x, y)$$

We will do two estimates, first a low estimate, then a high estimate.

$$V_{\text{min}} = \sum_i \Delta x \Delta y f(x, y)_{\text{min}} = (4)(3)[1 + 3 + 2 + 5] = 12(11) = 132$$

$$V_{\text{max}} = \sum_i \Delta x \Delta y f(x, y)_{\text{max}} = (4)(3)[5 + 9 + 9 + 15] = 12(38) = 456$$

Consider units:

$$f(x, y) = \frac{\text{larvae}}{m^2} \quad x = y = m$$

So, a volume represents $\frac{\text{larvae}}{m^2} \cdot m \cdot m = \text{larvae}$

Note that the difference between $V_{\text{max}}$ and $V_{\text{min}}$ is large in this case. This is because $\Delta x$ and $\Delta y$ were large. The estimate could be improved by using lower values of $\Delta x$ and $\Delta y$.

Another way to improve the estimate is to average $V_{\text{max}}$ and $V_{\text{min}}$.

$$\int_R f(x, y)\,dA \approx \frac{132 + 456}{2} = 294 \text{ larvae}$$

Average Value of a Function

Recall in single variable calculus, we found the average value of a function over an interval.

$$= \frac{1}{b - a} \int f(x)\,dx$$
Note that this represents an average value of f(x), not the average of $\int f(x)\,dx$!

**Extension to two variables**

$$\overline{f(x, y)} = \frac{1}{Area} \int_R f(x, y)\,dA$$

So in our recent example,

$$\overline{f(x, y)} = \frac{1}{Area} (294 \text{ larvae})$$

Area = (6m)(8m) = 48 m²

$$\overline{f(x, y)} = 6.125 \frac{\text{larvae}}{m^2}$$

This is the average density of mosquito larvae in this region of the swamp.
Positive and Negative Volumes

**Exercise 16.1.11**

- Determine the sign of the $\int_R 5x \, dA$ integrated over the region $R$ shown below.

  Note: $z = 5x$. In the right half of the unit circle below, $x > 0$ so therefore, $z$ is above the $xy$ plane. Hence the volume formed, and the integral would be positive.

![Diagram of a unit circle divided into two regions, L and R, with arrows indicating the direction of integration.

**Exercise 16.1.13**

However, if we integrated over the entire region $D$, $\int_D 5x \, dA = 0$ since $x < 0$ in $L$.

Note $dA$ is always positive, the sign of the integral is determined by $f(x,y)$. 

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