Lecture: Section 14.4

- Gradients and Directional Derivatives in a Plane
  
  o We know how to find the rate of change of a function in the direction parallel to the coordinate axes (i.e. by taking the partial derivative).
  
  o In this section we will learn how to find the derivative in an arbitrary direction. We will call this the directional derivative, which will be the rate of change in the direction of a unit vector \( \mathbf{u} \).
  
  o **Definition:** If \( \mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} \) is a unit vector, we define the directional derivative \( f_{\mathbf{u}} \) by:

\[
\begin{align*}
  f_{\mathbf{u}}(a, b) &= \text{Rate of change of } f \text{ in direction } = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h} \\
  &\text{of } \mathbf{u} \text{ at } (a, b).
\end{align*}
\]

  o **Ex. 14.4.4:** Example of directional derivative using a contour diagram.

  - Decide if directional derivative at \((-1, 1)\) in the direction of \(-i + j\) is positive, negative, or about zero.
Note that the vector $-\hat{i} + \hat{j}$ at (-1, 1) points in the direction of increasing values of $z$, so we would expect $f_{\vec{u}}$ to be positive.

- **Gradient and Directional Derivative**
  - Fortunately, we will not need to use the limit definition to find the directional derivative. We will use the gradient.
  - Here is the supporting development:

$$f_{\vec{u}}(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h} = \lim_{h \to 0} \frac{\Delta f}{h}$$

Recall:

$$\Delta f \equiv f_x(a, b) \Delta x + f_y(a, b) \Delta y$$

$$\Delta x = (a + hu_1 - a) = hu_1; \quad \Delta y = (b + hu_2 - b) = hu_2$$

$$\Delta f \approx \frac{f_x(a, b)hu_1 + f_y(a, b)hu_2}{h}$$

This approximation becomes exact as $h$ goes to zero, so we get:

$$f_{\vec{u}}(a, b) = f_x(a, b)u_1 + f_y(a, b)u_2$$

Where $f_x$ and $f_y$ are the partial derivatives of $f(x, y)$ and $u_1$ and $u_2$ are the components of the unit vector

- This is where **gradient** enters the picture.

- We see that we could obtain the directional derivative from the following dot product:

$$(f_x(a, b)\hat{i} + f_y(a, b)\hat{j}) \cdot (u_1 \hat{i} + u_2 \hat{j})$$

Where $f_x(a, b)\hat{i} + f_y(a, b)\hat{j}$ is called the gradient of $f(a, b)$, also written as, $\nabla f(a, b)$.

**Note:** The gradient is a vector.

- What does the gradient tell us?
Given: \( f_{\vec{u}} = \vec{\nabla} f \cdot \vec{u} \)

\[ = \|\vec{\nabla} f\| \|\vec{u}\| \cos \theta \]

But \( \|\vec{u}\| = 1 \) since \( \vec{u} \) is a unit vector

- So, we see that \( f_{\vec{u}} \) is maximum in the direction of the gradient, and max \( f_{\vec{u}} = \|\vec{\nabla} f\| \)
- Likewise, the minimum value of \( f_{\vec{u}} \) will be in a direction opposite of \( \vec{\nabla} f \) and be equal to \(-\|\vec{\nabla} f\|\)
- Note that since the direction of the gradient represents the direction of the maximum rate of change, it will be perpendicular to the contour, since this would be the shortest path to the next contour.

- **Geometric Properties of the Gradient Vector in a Plane**
  - The gradient’s direction is:
    - Perpendicular to contour of \( f \) through \((a, b)\)
    - In the direction of increasing \( f \)
  - The magnitude of the gradient vector is:
    - The maximum rate of change of \( f \) at that point.
    - Large when contours are close together and small when contours are far apart.

- **Examples:**
  - **14.4.8**
    - Estimate the gradient’s direction in Figure 14.31 at \((0, 2)\).

\[ \vec{\nabla} f \] is perpendicular to contours in the direction of increasing \( f \), so \( \vec{\nabla} f \) should point in the \( j \) direction.
14.4.18

- Find the gradient for \( z = xe^y = f(x, y) \)

\[
\frac{\partial z}{\partial x} = e^y \quad \text{and} \quad \frac{\partial z}{\partial y} = xe^y
\]

\[
\vec{\nabla} z = e^y \hat{i} + xe^y \hat{j}
\]

14.4.29

- Find \( \vec{\nabla} f \) at \((1, 2)\) for \( f(x, y) = x^2y + 7xy^3 \)

\[
\frac{\partial f}{\partial x} = 2xy + 7y^3 \Big|_{1,2} = 4 + 56 = 60
\]

\[
\frac{\partial f}{\partial y} = x^2 + 21xy^2 \Big|_{1,2} = 1 + 84 = 85
\]

\[
\vec{\nabla} f(1,2) = 60\hat{i} + 85\hat{j}
\]
14.4.36

\[ \vec{u} = \frac{1}{5} (3\hat{i} - 4\hat{j}) \] note: \[||\vec{u}|| = 1\]

\[ P = (1, 2) \quad f(x, y) = x^2 - y^2 \quad \frac{\partial f}{\partial x} = 2x \quad \frac{\partial f}{\partial y} = -2y \]

\[ f_{\vec{u}} = (2\hat{i} - 4\hat{j}) \cdot \frac{1}{5} (3\hat{i} - 4\hat{j}) = \frac{1}{5} (6 + 16) \]

\[ f_{\vec{u}} = \frac{22}{5} \]