MA 3610 R01
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(Secs 12.1-12.5, 13.3-13.4, 14.1-14.7, 15.1 & 15.3)

**General Directions:** This exam is closed book and closed notes. Calculators are allowed. Work must be shown to support correct answers for full credit.

1. Match the name of the surface with a plausible equation.

   a. (2 pts) \( z = y^2, \ x \text{ free to vary} \)
   
   b. (2 pts) \( x^2 + \frac{y^2}{4} + z^2 = 9 \)
   
   c. (2 pts) \( x^2 + y^2 = 4, z \text{ free to vary} \)
   
   d. (2 pts) \( z = -\sqrt{x^2 + y^2} \)

2. (5 pts) Write a linear function with the given points.

   \[
   f(x, y) = f(a, b) + m(x-a) + n(y-b)
   \]

   \[
   m = \frac{\Delta z}{\Delta x} = -\frac{1}{100}, \quad n = \frac{\Delta z}{\Delta y} = \frac{3}{10}
   \]

   \[(a, b) = (400, 10)\]

   \[
   f(x, y) = 0 + -\frac{1}{100} (x-400) + \frac{3}{10} (y-10)
   \]

   \[
   = -\frac{1}{100} x + 4 + \frac{3}{10} y - 3 = \left(\frac{1}{100} x + \frac{3}{10} y + 1\right)
   \]

3. (5 pts) (Sketch the level surface of the function \( g(x, y, z) = x^2 + y^2 + z \) when \( g \) is set equal to the constant 1.

\[
1 = x^2 + y^2 + z
\]

\[
z = 1 - (x^2 + y^2)
\]

(OVER)
4. Given the plane \( z - 5(x-2) = 3(5-y) \).

\[
\mathbf{n} = 5\hat{i} - 3\hat{j} - \hat{k}
\]

a. (3 pts) Find the normal vector to the plane.

b. (4 pts) Graph the plane on the given axes and indicate the values of the x-, y-, and z-intercepts.

\[
\begin{align*}
x \text{ int } &= -1 \\
y \text{ int } &= 5/3 \\
z \text{ int } &= 15 - 3x - 5y + 2z
\end{align*}
\]

5. Let \( \mathbf{v} = 2\hat{i} - 3\hat{j} - \hat{k} \) and \( \mathbf{w} = 5\hat{i} + 3\hat{j} - \hat{k} \).

\[
\frac{2}{2} = 10 - 9 + 1
\]

a. (3 pts) Find \( \mathbf{v} \cdot \mathbf{w} \).

Neither

b. (2 pts) Are the two vectors parallel, perpendicular or neither?

c. (3 pts) Find \( \mathbf{v} \times \mathbf{w} \). Calculators can be used only for checking.

\[
\begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
2 & -3 & -1 \\
5 & 3 & -1
\end{vmatrix}
= (2(-3) - (-1)(-1)) \hat{i} + (5(-1) - (-1)(5)) \hat{j} + (5(3) - 2(-1)) \hat{k}
= 6\hat{i} - 3\hat{j} + 17\hat{k}
\]

d. (2 pts) Find \( \tan \theta \) where \( \theta \) is the angle between \( \mathbf{v} \) and \( \mathbf{w} \).

\[
\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\sqrt{V \times W}}{V \cdot W} = \frac{\sqrt{V \cdot W \sin \theta}}{V \cdot W \cos \theta} = \frac{21}{2} = 11
\]

Note: \( \sqrt{V \cdot W} = \sqrt{V \cdot W} \cdot \sin \theta \)

\[V \cdot W = \sqrt{V \cdot W} \cdot \cos \theta\]

\[\sqrt{V \times W} = \sqrt{(b^2 + (-3)^2 + (2)^2)} = \sqrt{486} = 22.0\]
6. Find the following partial derivatives.

\[ S(3x + 2x)(3y + 2) \]  
\[ 2x e^{2xy} \]

a. (4 pts) \( z_x \) if \( z = (3xy + 2x)^5 \)

b. (4 pts) \( \frac{\partial f}{\partial y} \) if \( f(x, y) = e^{2xy} \)

c. (4 pts) \( \frac{\partial f}{\partial x} \bigg|_{(\frac{\pi}{3}, 1)} \) if \( f(x, y) = \frac{\sin(xy)}{y} \)

\[
\frac{\partial f}{\partial x} = \frac{1}{y} \left[ \cos(xy) \right] y = \cos(xy) \bigg|_{(\frac{\pi}{3}, 1)} = \cos(\frac{\pi}{3}) = \frac{1}{2}
\]

d. (4 pts) \( g_{xy} \) if \( g(x, y) = 3x^2y + 2x \ln y \)

\[
g_x = 6xy + 2 \ln y
\]

\[
(g_x)_y = 6x + \frac{2}{y}
\]

7. Let \( f(x, y) = x^2 + y^3 \)

\[
f_{(x,y)} = 9 + 2(x-1) + 12(y-2)
\]

a. (4 pts) Find the local linearization of \( f(x, y) \) at the point \( (1,2) \).

\[
f_x(1,2) = 2 \quad f_y(1,2) = 12
\]

\[
f(x, y) = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2)
\]

\[
f(x, y) = 9 + 2(x-1) + 12(y-2)
\]

\[
f(1,2) = 9
\]

b. (2 pts) Estimate \( f(1.04, 1.98) \) using the linearization.

\[
f(1.04, 1.98) = 9 + 2(0.04) + 12(-0.02)
\]

\[
= 9 + 0.08 + -0.24
\]

\[
= 8.84
\]
8. Let \( f(x,y) = x^2 + \ln y \) and let point \( P = (3,1) \).

\[
\nabla f(3,1) = 6\hat{i} + \hat{j}
\]

a. (4 pts) Find \( \nabla f \) at the point \( P \).
\( \nabla f = 2x\hat{i} + \frac{1}{y}\hat{j} \)

b. (2 pts) In what direction from the point \( P \) is \( f \) increasing the fastest?

\( \sqrt{2} = \frac{5\sqrt{2}}{2} \)

c. (4 pts) Find the directional derivative of \( f \) at the point \( P \) in the direction of \( \hat{u} = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j} \).

unit vector

\[

\frac{\nabla f \cdot \hat{u}}{||\hat{u}||} = (6\hat{i} + \hat{j}) \cdot \frac{1}{\sqrt{2}}(\hat{i} - \hat{j}) = \frac{5}{\sqrt{2}}
\]

9. Let \( w = f(x,y,z) = x^2 + 2y + z \) where \( x = u^2 + v^2 \), \( y = u^2v^2 \) and \( z = u \).

a. (3 pts) Draw a tree diagram to the right expressing the relationships between the variables.

\[
8u^2v + 4v^3
\]

b. (4 pts) Find \( \frac{\partial f}{\partial v} \).
\[
\frac{\partial f}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v}
\]
\[
= (2x)(2v) + (2)(2u^2v)
\]
\[
= 2(u^2 + v^2)(2v) + 4u^2v
\]
\[
= 4u^2v + 4v^3 + 4u^2v
\]
\[
= 8u^2v + 4v^3
\]
10. Use the level curve of the function \( z = f(x,y) \) to decide the sign (+, -, 0) of each of the following partial derivatives.

a. (2 pts) \( f_x \)

b. (2 pts) \( f_{xx} \)

c. (2 pts) \( f_y \)

d. (2 pts) \( f_{yy} \)

11. (6 pts) Find the critical points of the function \( f(x, y) = x^3 + y^2 - 3xy \) and classify each as a local max, a local min, a saddle point or none of these.

\[
\begin{align*}
\nabla f &= \left( 3x^2 - 3y, 2y - 3x \right) \\

f_x &= 3x^2 - 3y \\
f_y &= 2y - 3x \\
x^2 &= y \\
y &= \frac{3}{2}x \\
x^2 - \frac{3}{2}x &= 0 \\
x(x - \frac{3}{2}) &= 0 \\
x = 0, \frac{3}{2} \\
y = 0, \frac{3}{2} \\

\begin{array}{|c|c|}
\hline
\text{CP's} & \\
\hline
x = 0, \frac{3}{2} & \\
y = 0, \frac{3}{2} & \\
\hline
\end{array}
\]

\[ D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & -3 \\ -3 & -2 \end{vmatrix} = 12x - 9 \]

\[
D(0,0) = -9 < 0 \\
f_{xx} = 6 \text{ saddle point} \\
D \left( \frac{3}{2}, \frac{9}{4} \right) = 6 \left( \frac{3}{2} \right) - 9 > 0 \\
f_{xx} = 6 \left( \frac{3}{2} \right) \text{ local minimum} \\
f_{xx} > 0
\]

Critical Points | Classification
\( (x, y) = (0, 0) \) | Saddle pt
\( (x, y) = \left( \frac{3}{2}, \frac{9}{4} \right) \) | local minimum

(OVER)
12. Given the contour diagram to the right. Identify each labeled point as a local minimum, a local maximum, a saddle point or none of these.

- None a. (2 pts) point A
- Local Max. b. (2 pts) point B
- Saddle c. (2 pts) point C

13. (6 pts) Use the method of Lagrange multipliers to find the maximum value of the function
\[ f(x, y) = -3x^2 - 2y^2 \]
subject to the constraint that x and y lie on the parabola
\[ y - x^2 = -2. \]

Find \( \nabla f = \lambda \nabla g \)

\[ -6x \hat{i} - 4y \hat{j} = \left( -2x \hat{i} + \hat{j} \right) \lambda \]

\[ -6x = -2 \lambda x \]
\[ -4y = \lambda \]
\[ -6x + 2\lambda x = 0 \]
\[ -4y = 3 \]
\[ \lambda x (\lambda - 3) = 0 \]
\[ \lambda = 3 \]
\[ x = 0 \]

Substitute \( x = 0, y = -3/4 \) into constraint.

\[ f(x, y) = -3x^2 - 2y^2 \]
\[ y - x^2 = -2 \]
\[ y = -2 \]
\[ x = 0 \Rightarrow y = -2 \]
\[ y = -2 \]
\[ x^2 = \frac{5}{4} \]
\[ x = \pm \frac{\sqrt{5}}{2} \]

Max value of \( f = \frac{-3\sqrt{5}}{8} \) and occurs at \((x, y) = \left( \frac{\sqrt{5}}{2}, -\frac{3}{4} \right) \)