1.) Find the unit normal vector to the plane which contains the points $A = (1,1,1)$, $B = (2,3,2)$ and $C = (1,0,0)$. Show all work. (15 pts).

\[
\begin{align*}
1 &= m+n+c \\
2 &= 2m+3n+c \\
0 &= m+0n+c
\end{align*}
\]

\[
\begin{align*}
v &= i + 2j + k & (3) \\
\bar{w} &= 0i - j - k & (3)
\end{align*}
\]

\[
\bar{v} \times \bar{w} = \bar{n} = -i + j - k & (5)
\]

\[
\begin{align*}
\bar{u} &= \frac{\bar{n}}{|\bar{n}|} = \frac{1}{\sqrt{3}} \left[ -i + j - k \right] & (4)
\end{align*}
\]

2.) Find the directional derivative of the function $f(x,y) = xy^2 + xe^x$ along a line extending from the origin to the point $(1,1)$. Show all work. No calculator solutions. (15 pts)

\[
\begin{align*}
f_x &= 2xy + xe^x + e^x & (5) \\
f_y &= x^2
\end{align*}
\]

\[
\nabla f = \left( 2xy + xe^x + e^x \right) \bar{i} + (x^2) \bar{j} & (3)
\]

\[
\bar{u} = \frac{1}{\sqrt{2}} (\bar{i} + \bar{j})
\]

\[
\nabla f \cdot \bar{u} = \frac{1}{\sqrt{2}} \left[ 2xy + xe^x + e^x + x^2 \right] & (2)
\]
3.) Match the proper equation with each surface shown below. (12 pts)

\[
\begin{align*}
  x &= \sqrt{y} & \quad z &= -y^2 \\
  x + z &= 4 & \quad x &= \sqrt{y}
\end{align*}
\]

\[\text{[Diagrams of surfaces]}\]

\[
\begin{align*}
  &\text{a.) } z = y^2 & \quad &\text{b.) } z = x^2 \\
  &\text{c.) } z = -x^2 & \quad &\text{d.) } z = -y^2 \\
  &\text{e.) } z = x^2 - y^2 & \quad &\text{f.) } z = x^2 + y^2 \\
  &\text{g.) } z = \sqrt{x^2 + y^2} & \quad &\text{h.) } z = -(x^2 + y^2) \\
  &\text{i.) } 4 = z^2 + y^2 & \quad &\text{j.) } 4 = x^2 + y^2 \\
  &\text{k.) } 4 = x^2 + z^2 & \quad &\text{l.) } z = 2x^2 + y^2 \\
  &\text{m.) } z = x^2 + 2y^2 & \quad &\text{n.) } z = -\sqrt{x^2 + y^2} \\
  &\text{o.) None of the above}
\end{align*}
\]

- major axis \( \sqrt{c} \) along \( y \)
- minor axis \( \sqrt{a} \) along \( x \)
4.) The temperature (Deg C) in a process vessel is given as a function of position coordinates x, y and z (meters) by the equation below. Assume that $0 \leq z \leq 2$.

$$T(x, y, z) = 25(x^2 + y^2)$$

(a.) What do the level surfaces of $T(x, y, z)$ look like?

A family of concentric cylinders.

(b.) Plot the level surface corresponding to a temperature of 100 Deg C on the axes shown to the right. Show and define the intercepts with the positive x- and y-axes.

$$100 = z \leq (x^2 + y^4)$$

$$x^2 + y^4 = 4$$

(c.) Find the gradient of $T(x, y, z)$ at the point (-2,2,1).

$$\nabla T = 50x \hat{i} + 50y\hat{j} = -100\hat{i} + 100\hat{j}$$

(d.) What are the units of the gradient?

$^\circ$C/m

(e.) What is the magnitude of the gradient at the point (-2,2,1)?

$$|\nabla T| = \sqrt{100^2 + 100^2} = \sqrt{20000} = 141.4$$

(f.) Sketch the gradient vector on the figure in Part B. \(\perp\) to surface at (-2,2,1)

(g.) How would the rate of change of temperature at the point (-2,2,1) in the $\hat{i}$ direction compare with that in the direction of the gradient? Explain.

It would be $\leq$ since it is max in direction of $\nabla T$. Actually $f_{\hat{i}} = -100\hat{i}$ so magnitude = 100 vs $141\sqrt{71}$. 
\[
0 = x(2, 1, 0) = (2)(0)(0) \quad z(2, 1, 0) = 0
\]

\[
y(2, 1, 0) = (2)
\]

5.) Let \( u = x^2y + xz^2 \) where \( x = rte^s \), \( y = rs \), and \( z = rs[\sin(t)] \). \((15 \text{ pts})\)

a.) Represent the relationships between the variables using a tree diagram.

\[
\frac{\partial x}{\partial s} = rte^s \quad \frac{\partial y}{\partial s} = r \quad \frac{\partial z}{\partial s} = \sin(t) \cdot r
\]

b.) Find \( \frac{\partial u}{\partial s} \) and evaluate it at \( r = 2 \), \( s = 1 \) and \( t = 0 \).

\[
\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial s}
\]

\[
= (2xy + 2z^2)(rte^s) + (x^2)(r) + (2xz)(r \sin(t))
\]

\[
\frac{\partial u}{\partial s} = 2(0)(2)(0)(1) + (1)^2 + (2)(2)(0)(0)
\]

\[
= 0 + 1 + 0 = 0
\]

6.) Use differentials to estimate the volume of material required to manufacture a closed can that is 4 cm in radius and 10 cm in height and 0.04 cm thick. The can is in the shape of a right circular cylinder. Note that the volume (\( V \)) of a right circular cylinder is \( V = \pi r^2h \), where \( r \) and \( h \) are the dimensions of the radius and height, respectively. **Show all work.** \((12 \text{ pts})\)

\[\text{d}V = \pi r^2h \]

\[\text{d}V = \pi (2r)h \text{dr} + \pi r^2 \text{dh}
\]

\[\text{dh} = 0.08 \quad \text{dr} = 0.04
\]

\[\text{d}V = \pi (2)(4)(10)(0.04) + \pi (4)^2 (0.08)
\]

\[= 3.2\pi + 1.28\pi = 4.48\pi
\]

\[= 14.1 \text{ cm}^3
\]
7.) For \( f(x, y) = x y^3 + (2x + 1)^3 y \) answer the following questions. Show all work. (15 pts).

a.) Find the linear approximation \( L(x, y) \) near the point \((1,1)\).

\[
L(a, b) = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b)
\]

\[
f_x = y^2 + 3(2x+1)^2, \quad f_y = 3xy^2 + (2x+1)^3
\]

\[
f_x(1,1) = 1 + 3(3)^2(2)(1) = 1 + 54 = 55
\]

\[
f_y(1,1) = 3(1)^2 + (2(1)+1)^3 = 3 + 27 = 30
\]

\[
f(1,1) = 1 + (3)^3 = 28
\]

\[
L(a, b) = 28 + 55(x-1) + 30(y-1) = 55x + 30y - 57
\]

b.) Use the approximation above to estimate \( f(1.05, 1.05) \)

\[
L(1.05, 1.05) = 28 + 55(0.05) + 30(0.05) = 32.25
\]

\[
f(1.05, 1.05) \approx 32.496
\]

c.) Find the equation of the tangent plane to the surface at the point \((1,1)\).

\[
TP = Z = L(x, y)
\]

\[
z = f_x(x-a) + f_y(y-b) + z_0
\]

\[
z = 28 + 55(x-1) + 30(y-1) = L(a, b)
\]

\[
z = 28 + 55x - 55 + 30y - 30 = \frac{-55}{28}
\]

\[
55x + 30y - z = 57
\]