1. Match the following functions with their graphs. (2 pts per answer)

   a. $10 = x + 2y + 3z$

   b. $z = -y^2$, $x$ is a free variable

   c. $z = \sqrt{x^2 + y^2}$

   d. $z = x^2 - y^2$

   e. $x^2 + y^2 = 4$, $z$ is a free variable

2. (10 pts) Find an equation for a linear function with the given contour diagram.
3. (10 pts) Find an equation of a plane perpendicular to the vector \(5\hat{i} + \hat{j} - 2\hat{k}\) and passing through the point \((0, 1, -1)\).

4. Let \(\vec{v} = 3\hat{i} + 2\hat{j} - 2\hat{k}\) and \(\vec{w} = 4\hat{i} - 3\hat{j} + \hat{k}\), find

a. \(\vec{v} \cdot \vec{w}\)  
(2 pts)

b. \(\vec{v} \times \vec{w}\)  
(2 pts)

c. A unit vector in the same direction as \(\vec{v}\).  
(2 pts)

d. The angle (to the nearest degree) between \(\vec{v}\) and \(\vec{w}\).  
(2 pts)

e. The component of \(\vec{v}\) in the direction of \(\vec{w}\).  
(2 pts)
5. The quantity, \( Q \), of beef purchased at a store in kilograms per week, is a function of the price of beef, \( b \), and the price of chicken, \( c \), both in dollars per kilogram.

a. (6 pts) Do you expect \( \frac{\partial Q}{\partial c} \) to be positive or negative? Explain why below.

positive/negative

b. (4 pts) Interpret the statement \( \frac{\partial Q}{\partial b} = -213 \).

6. Find the following partial derivatives.

a. \( g_x \) if \( g(x,y) = y e^{xy} \)

(3 pts)

b. \( \frac{\partial V}{\partial r} \) if \( V = \frac{4}{3} \pi r^2 h \)

(3 pts)

c. \( f_y(1,2) \) if \( f(x,y) = x^2 + 3x^2y - 2y^2 \)

(4 pts)
7. Let \( f(x,y) = x^2 + \ln y + z^3 \)

a. Find the directional derivative of \( f \) at \( P = (1,3,2) \) in the direction of \( \hat{v} = 3\hat{i} - 2\hat{j} + \hat{k} \).

(6 pts)

b. What is the direction of greatest increase of \( f \) at \( P \)?

(4 pts)

8. For the function \( w = f(x,y,z) = x^2 - y^2 + z^2 \), one level is \( 4 = x^2 - y^2 + z^2 \).

a. (5 pts) Find a vector normal to the level surface at the point \((-1, 1, 2)\).

b. (5 pts) Find an equation for the tangent plane to the level surface at the point \((-1, 1, 2)\).

9. Let \( z = xe^y \), \( x = u^2 + v^2 \) and \( y = u^2 - v^2 \)

a. (5 pts) Draw a tree diagram to the right expressing the relationships between the variables.

b. Find \( \frac{\partial z}{\partial u} \). Express your answer in terms of \( u \) and \( v \) only.

(5 pts)
General Directions: Your exam will be closed book and closed notes. Calculators are allowed. Work must be shown to support correct answers for full credit.

1. Match the following functions with their graphs. (2 pts per answer)

   III. a. \(10 = x + 2y + 3z\)

   IV. b. \(z = -y^2, x\) is a free variable

   VI. c. \(z = \sqrt{x^2 + y^2}\)

   V. d. \(z = x^2 - y^2\)

   II. e. \(x^2 + y^2 = 4, z\) is a free variable

2. (10 pts) Find an equation for a linear function with the given contour diagram.

\[
f(x, y) = \frac{4-0}{-1-3} \cdot \frac{-4}{2} = -2
\]

\[
m = \frac{\Delta f}{\Delta x} = \frac{12-6}{1--1} = \frac{6}{2} = 3
\]

\[
z = c + mx + ny
\]

\[
(x, y) = (-3, -6)
\]

\[
s = -6 = c - 2(0) + 3(3)
\]

\[
-6 = c - 9
\]

\[
3 = c
\]
3. (10 pts) Find an equation of a plane perpendicular to the vector \( 5\mathbf{i} + \mathbf{j} - 2\mathbf{k} \) and passing through the point \((0, 1, -1)\).

\[
\begin{align*}
5x + y - 2z &= 3 \\
5(0) + 1 - 2(-1) &= 3 \\
1 + 2 &= 3 \\
\mathbf{n} &= 5\mathbf{i} + \mathbf{j} - 2\mathbf{k} \\
\mathbf{p} &= (0, 1, -1) \\
\mathbf{n} \cdot \mathbf{p} &= 0
\end{align*}
\]

4. Let \( \mathbf{v} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} \) and \( \mathbf{w} = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k} \), find

- \( \mathbf{v} \cdot \mathbf{w} = 3(4) + 2(-3) - 2(1) = 12 - 6 - 2 = 4 \)
- \( \mathbf{v} \times \mathbf{w} = \begin{vmatrix} 
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
3 & 2 & -2 \\
4 & -3 & 1 
\end{vmatrix} = \mathbf{i}(2 - 6) + \mathbf{j}(-8 - 3) + \mathbf{k}(-9 - 8) = -4\mathbf{i} - 11\mathbf{j} - 17\mathbf{k} \)
- A unit vector in the same direction as \( \mathbf{v} \) is \( \mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||} = \frac{4\mathbf{i} - 3\mathbf{j} + \mathbf{k}}{\sqrt{16 + 9 + 1}} = \frac{4}{\sqrt{126}}\mathbf{i} - \frac{3}{\sqrt{126}}\mathbf{j} + \frac{1}{\sqrt{126}}\mathbf{k} \)
- The angle (to the nearest degree) between \( \mathbf{v} \) and \( \mathbf{w} \) is \( \Theta = \cos^{-1}\left(\frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{v}|| \cdot ||\mathbf{w}||}\right) = \cos^{-1}\left(\frac{4}{\sqrt{126} \cdot \sqrt{17}}\right) = 79.03^\circ \approx 79^\circ \)
- The component of \( \mathbf{v} \) in the direction of \( \mathbf{w} \) is \( \mathbf{v}_{\parallel} = \left(\frac{\mathbf{v} \cdot \mathbf{u}}{||\mathbf{u}||}\right) \mathbf{u} = \left(\frac{4}{\sqrt{126}}\mathbf{i} - \frac{3}{\sqrt{126}}\mathbf{j} + \frac{1}{\sqrt{126}}\mathbf{k}\right) \left(\frac{4\mathbf{i} - 3\mathbf{j} + \mathbf{k}}{\sqrt{126}}\right) = \frac{16}{126} \mathbf{i} - \frac{12}{126} \mathbf{j} + \frac{4}{126} \mathbf{k} \approx \frac{8}{13} \mathbf{i} - \frac{6}{13} \mathbf{j} + \frac{2}{13} \mathbf{k} \)
5. The quantity, $Q$, of beef purchased at a store in kilograms per week, is a function of the price of beef, $b$, and the price of chicken, $c$, both in dollars per kilogram.

- (6 pts) Do you expect $\frac{\partial Q}{\partial c}$ to be positive or negative? Explain why below.

  If the price of chicken goes up, people will decrease the amount of beef they buy.

- (4 pts) Interpret the statement $\frac{\partial Q}{\partial b} = -213$.

  If the price of beef rose by one dollar, the store would sell approximately 213 fewer kilograms of beef that week.

6. Find the following partial derivatives.

- (3 pts) $g_x$ if $g(x,y) = ye^{xy}$

  $g_x = ye^{xy}$

- (3 pts) \frac{\partial V}{\partial r}$ if $V = \frac{4}{3} \pi r^2 h$

  $\frac{\partial V}{\partial r} = \frac{4}{3} \pi (2r)h = \frac{8}{3} \pi rh$

- (4 pts) $f_y(1,2)$ if $f(x,y) = x^3 + 3x^2y - 2y^2$

  $f_y = 0 + 3x^2 - 4y$

  $f_y(1,2) = 3(1)^2 - 4(2) = 3 - 8 = -5$
7. Let \( f(x, y) = x^2 - y^2 \)

\[
\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x \\ -2y \end{pmatrix}
\]

\[
a. \text{ Find the directional derivative, } f_\mathbf{a}(1, 2) \text{ in the direction of } \mathbf{a} = \frac{3i - 4j}{5}
\]

\[
f_\mathbf{a}(1, 2) = (2x \mathbf{i} - 2y \mathbf{j}) \cdot \left( \frac{3}{5} \mathbf{i} - \frac{4}{5} \mathbf{j} \right)
\]

\[
\frac{2}{5} + \frac{16}{5} = \frac{22}{5} = 4 \frac{2}{5}
\]

\[
b. \text{ What is the direction of greatest increase of } f \text{ at } P = (1, 2)?
\]

\[
\nabla f(1, 2) = 2 \mathbf{i} - 4 \mathbf{j}
\]

8. For the function \( w = f(x, y, z) = x^2 - y^2 + z^2 \), find an equation for the tangent plane to the level surface at the point \((-1, 1, 2)\) using the following formula for a tangent plane:

\[
f_x(a, b, c)(x-a) + f_y(a, b, c)(y-b) + f_z(a, b, c)(z-c) = 0
\]

\[
\nabla f(1, 1, 2) = 2(1) \mathbf{i} - 2(1) \mathbf{j} + 2(2) \mathbf{k}
\]

\[
\mathbf{n} = -2 \mathbf{i} + 2 \mathbf{j} + 4 \mathbf{k}
\]

9. Let \( z = xe^y \), \( x = u^2 + v^2 \) and \( y = u^2 - v^2 \)

a. (5 pts) Draw a tree diagram to the right expressing the relationships between the variables.

\[
\frac{2u \ e^{y^2}}{(1 + u^2 + v^2)}
\]

b. Find \( \frac{\partial z}{\partial u} \). Express your answer in terms of \( u \) and \( v \) only.

\[
\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}
\]

\[
= e^y \ 2u + xe^y \ 2u
\]

\[
= 2ue^y (1 + x) = 2ue^y \left( 1 + u^2 + v^2 \right)
\]