Symmetric Functions and Quasisymmetric Functions

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Outline

1. Symmetric Functions

2. NSym and QSym

3. Categorification of the Heisenberg Double

4. Application: QSym is free over Sym
Symmetric Functions

Definition

- $R$: commutative ring with identity
- $x = (x_1, x_2, \cdots)$: set of indeterminates
- $n$: nonnegative integer

A homogeneous symmetric function of degree $n$ is a formal power series $f(x) = \sum_\alpha c_\alpha x^\alpha$ where

- $\alpha$ ranges over all weak compositions $\alpha = (\alpha_1, \alpha_2, \cdots)$ of $n$,
- $c_\alpha \in R$,
- $x^\alpha$ stands for the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots$,
- $f(x_{w(1)}, x_{w(2)}, \cdots) = f(x_1, x_2, \cdots)$ for every permutation $w$ of the positive integers.
Symmetric Functions

Definition
Let $\Lambda^n_R$ be the set of all homogeneous symmetric functions of degree $n$.

$$\Lambda_R = \Lambda^0_R \oplus \Lambda^1_R \oplus \cdots$$

is a commutative, unital, graded $R$-algebra.

Bases for $\Lambda^n_Q$
- Monomial symmetric functions $\{m_\lambda : \lambda \vdash n\}$
- Elementary symmetric functions $\{e_\lambda : \lambda \vdash n\}$
- Complete homogeneous symmetric functions $\{h_\lambda : \lambda \vdash n\}$
- Power sum symmetric functions $\{p_\lambda : \lambda \vdash n\}$
- Schur functions $\{s_\lambda : \lambda \vdash n\}$
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Symmetric Functions Over Integers

\[ \Lambda_\mathbb{Z} = \text{Sym} = \mathbb{Z}[e_1, e_2, \cdots] \subset \mathbb{Z}[x_1, x_2, \cdots] \]

- \( e_1 = x_1 + x_2 + \cdots \)
- \( e_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 + \cdots \)
- \( e_n = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n} \)

Sym as a Hopf algebra

- \( \triangle : \text{Sym} \rightarrow \text{Sym} \otimes \text{Sym}, \quad e_n \mapsto \sum_{i+j=n} e_i \otimes e_j \)
- \( \epsilon : \text{Sym} \rightarrow \mathbb{Z}, \quad e_n \mapsto 0, \quad n \geq 1 \)

Connection to Representation Theory

- (Geissinger 1977) \( \text{Sym} \cong \bigoplus_{n=0}^{\infty} K_0(\mathbb{C}[S_n]\text{-mod}) \)
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Connection to Representation Theory

- (Geissinger 1977) \( \text{Sym} \cong \bigoplus_{n=0}^{\infty} \mathcal{K}_0(\mathbb{C}[S_n]\text{-mod}) \)
Duality of Sym

Bilinear Form on Sym

- Define \( \langle \cdot , \cdot \rangle : \text{Sym} \times \text{Sym} \rightarrow \mathbb{Z} \) by \( \langle m_\lambda, h_\mu \rangle = \delta_{\lambda,\mu} \) for \( \lambda, \mu \in \mathcal{P} \).
- \( \langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu} \)

Bilinear Form on Sym \( \otimes \) Sym

- Define \( (\cdot , \cdot) : \text{Sym} \otimes \text{Sym} \times \text{Sym} \otimes \text{Sym} \rightarrow \mathbb{Z} \) by \( (x \otimes y, x' \otimes y') = \langle x, x' \rangle \langle y, y' \rangle \).
- \( (x \otimes y, \triangle(z)) = \langle \nabla(x \otimes y), z \rangle \)

Sym \( \cong \) Sym*

- \( \text{Sym}^* = \bigoplus_{n \in \mathbb{N}} (\Lambda^n_{\mathbb{Z}})^* : \) graded dual of Sym
- \( \Phi : \text{Sym} \cong \text{Sym}^* \) by \( \Phi(x)(y) = \langle x, y \rangle \).
Duality of \( \text{Sym} \)

**Bilinear Form on \( \text{Sym} \)**

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- \( \text{Sym}^* = \bigoplus_{n \in \mathbb{N}} (\Lambda^n_{\mathbb{Z}})^* \): graded dual of \( \text{Sym} \)
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Bilinear Form on $\text{Sym} \otimes \text{Sym}$

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- $(x \otimes y, \triangle(z)) = \langle \bigtriangledown(x \otimes y), z \rangle$

Sym $\cong \text{Sym}^*$

- $\text{Sym}^* = \bigoplus_{n \in \mathbb{N}} (\wedge^n_{\mathbb{Z}})^* :$ graded dual of Sym
- $\Phi : \text{Sym} \cong \text{Sym}^*$ by $\Phi(x)(y) = \langle x, y \rangle$. 
Noncommutative Symmetric Functions

Definition

\[ \text{NSym} = \mathbb{Z}\langle h_1, h_2, \cdots \rangle: \text{free algebra} \]

NSym as a Hopf algebra

- \( \triangle : \text{NSym} \to \text{NSym} \otimes \text{NSym}, \quad h_n \mapsto \sum_{i+j=n} h_i \otimes h_j \)
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Connection to Representation Theory

- (Duchamp, Krob, Leclerc, Thibon, Ung, 1996)

\[ \text{NSym} \cong \bigoplus_{n=0}^{\infty} \mathcal{K}_0(H_n(0)\text{-pmod}) \]
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Quasisymmetric Functions

Definition (Gessel 1984)

$\text{QSym} \subset \mathbb{Z}[[x_1, x_2, \cdots]]$ consisting of shift invariant formal power series of bounded degree, i.e., $f \in \text{QSym}$ if and only if

$$\text{coeff of } x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} \text{ in } f = \text{coeff of } x_{i_1}^{n_1} x_{i_2}^{n_2} \cdots x_{i_k}^{n_k} \text{ in } f$$

for all $0 < i_1 < i_2 < \cdots < i_k$ and $n_1, n_2, \cdots, n_k \in \mathbb{N}$.

Example

- $\sum_{i<j} x_i^2 x_j$ quasisymmetric, not symmetric.
- $\sum_{i<j} x_i x_j^5$ quasisymmetric, not symmetric.
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Quasisymmetric Functions

Additive Basis for $QSym$

- $M_{\alpha} = \sum_{i_1<\ldots<i_k} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$, where $\alpha \in \text{Comp}(n)$.

$QSym$ as a Hopf algebra

- Multiplication: overlapping shuffles
- Comultiplication: cut

Duality of $NSym$ and $QSym$

- Define $\langle \cdot, \cdot \rangle : NSym \times QSym \to \mathbb{Z}$ by $\langle h_{\alpha}, M_{\beta} \rangle = \delta_{\alpha,\beta}$.
- $(\cdot, \cdot) : NSym \otimes NSym \times QSym \otimes QSym \to \mathbb{Z}$
- $(\triangle(h_{\alpha}), M_{\beta} \otimes M_{\gamma}) = \langle h_{\alpha}, \nabla(M_{\beta} \otimes M_{\gamma}) \rangle$
- $QSym \cong NSym^*$
Quasisymmetric Functions

Additive Basis for $\text{QSym}$

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- Define $\langle \cdot , \cdot \rangle : \text{NSym} \times \text{QSym} \rightarrow \mathbb{Z}$ by $\langle h_{\alpha}, M_{\beta} \rangle = \delta_{\alpha,\beta}$.
- $(\langle \cdot , \cdot \rangle : \text{NSym} \otimes \text{NSym} \times \text{QSym} \otimes \text{QSym} \rightarrow \mathbb{Z}$
- $(\triangle(h_{\alpha}), M_{\beta} \otimes M_{\gamma}) = \langle h_{\alpha}, \nabla(M_{\beta} \otimes M_{\gamma}) \rangle$
- $\text{QSym} \cong \text{NSym}^*$
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- \( (\cdot, \cdot) : \text{NSym} \otimes \text{NSym} \times \text{QSym} \otimes \text{QSym} \rightarrow \mathbb{Z} \)
- \( (\triangle(h_\alpha), M_\beta \otimes M_\gamma) = \langle h_\alpha, \nabla(M_\beta \otimes M_\gamma) \rangle \)
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Polynomial Freeness of $\text{QSym}$

**Ditters Conjecture 1972**

The algebra $\text{QSym}$ is a free commutative algebra over the integers.

**Hazewinkel 2001, 2002**

- Ditters Conjecture is proved.
- An explicit free commutative polynomial basis is constructed.

**$\text{QSym}$ is free over $\text{Sym}$**

- $E = \{e_n(\alpha) \mid \alpha \in e\text{LYN}, n \in \mathbb{N}\}$: free polynomial basis for $\text{QSym}$.
- $E$ contains the elementary symmetric functions.
Polynomial Freeness of $QSym$

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The Heisenberg Double

Definition (Dual Pair)

\((H^+, H^-)\) is a dual pair of Hopf algebras if

- \(H^\pm\) are graded connected Hopf algebras,
- we have a perfect Hopf pairing \(\langle \cdot, \cdot \rangle : H^- \times H^+ \to R\).

Via this pairing, identify \(H^\pm\) with the grade dual of \(H^\mp\).

Definition (Heisenberg Double)

The Heisenberg double of \(H^+\) is the algebra \(\mathfrak{h} = \mathfrak{h}(H^+, H^-)\) given by

- \(\mathfrak{h} = H^+ \otimes H^-\) as \(R\)-modules.
  We write \(a \# x\) for \(a \otimes x\), viewed as an element of \(\mathfrak{h}\).
- Multiplication is given by:
  \((a \# x)(b \# y) = \sum_{(x)} a^R x_{(1)}^* (b) \# x_{(2)} y = \sum_{(x), (b)} \langle x_{(1)}, b_{(2)} \rangle a b_{(1)} \# x_{(2)} y\).
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Fock Space Representation

Definition (Fock Space Representation)
The algebra $\mathfrak{h}$ has a natural representation on $H^+$ given by

$$(a \sharp x)(b) = a^R x^*(b), \quad a, b \in H^+, \ x \in H^-.$$

Stone-von Neumann Type Theorem (Savage, Yacobi 2015)

- The representation $\mathcal{F}$ is faithful.
- If $R$ is a field, then $\mathcal{F}$ is irreducible.
- Any representation of $\mathfrak{h}$ generated by a lowest weight vacuum vector is isomorphic to $\mathcal{F}$. 

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Sym and QSym

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Example

\[
\text{Sym} \cap \text{NSym} \leftrightarrow \text{QSym} \\
\downarrow \\
\text{Sym}
\]

Heisenberg Algebra \( h = h(\text{Sym}, \text{Sym}) \)
- \( p_1, p_2, \cdots \): the power sums in \( H^+ = \text{Sym} \).
- \( p_1^*, p_2^*, \cdots \): the power sums in \( H^- = \text{Sym} \).
- \( p_m p_n = p_n p_m, \quad p_m^* p_n^* = p_n^* p_m^*, \quad p_m^* p_n = p_n p_m^* + m\delta_{m,n} \).

Quasi-Heisenberg Algebra \( q = h(\text{QSym}, \text{NSym}) \)
- Fock space representation: natural action on \( \text{QSym} \).
- \( q_{\text{proj}} \): subalgebra generated by \( \text{Sym} \subset \text{QSym} \) and \( \text{NSym} \).
Example

Sym \cap NSym \leftrightarrow QSym

Heisenberg Algebra \mathfrak{h} = \mathfrak{h}(\text{Sym, Sym})

- \(p_1, p_2, \cdots\): the power sums in \(H^+ = \text{Sym}\).
- \(p^*_1, p^*_2, \cdots\): the power sums in \(H^- = \text{Sym}\).
- \(p_m p_n = p_n p_m, \quad p^*_m p^*_n = p^*_n p^*_m, \quad p^*_m p_n = p_n p^*_m + m \delta_{m,n}\).

Quasi-Heisenberg Algebra \(\mathfrak{q} = \mathfrak{h}(\text{QSym, NSym})\)

- Fock space representation: natural action on QSym.
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Heisenberg Algebra \( \mathfrak{h} = \mathfrak{h}(\text{Sym}, \text{Sym}) \)

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Categorification

Goal
To categorify Heisenberg doubles and their Fock space representations.

What is categorification?
Suppose $M$ is a module for a ring $R$. We would like to find an abelian category $\mathcal{M}$ such that

$$K_0(\mathcal{M}) \xrightarrow{\phi} M$$ (as $\mathbb{Z}$-modules),

where $K_0(\mathcal{M})$ is the Grothendieck group of $\mathcal{M}$. 
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where $\mathcal{K}_0(\mathcal{M})$ is the Grothendieck group of $\mathcal{M}$. 
Categorification

For each $r \in R$ (or, for those $r$ in a fixed generating set), we want an exact endofunctor $F_r$ of $\mathcal{M}$ such that we have a commutative diagram:

$$
\begin{array}{ccc}
\mathcal{K}_0(\mathcal{M}) & \xrightarrow{[F_r]} & \mathcal{K}_0(\mathcal{M}) \\
\phi \downarrow & & \phi \downarrow \\
M & \xrightarrow{r} & M
\end{array}
$$

Here $[F_r]$ denotes the map induced by $F_r$ on $\mathcal{K}_0(\mathcal{M})$.

We would also like isomorphisms of functions lifting the relations of $R$. For example, suppose we have a relation in $R$: $rs = 2sr + 3$. Then we would like isomorphisms of functors $F_r \circ F_s \cong (F_s \circ F_r) \oplus 2 \oplus \text{Id} \oplus 3$. 
Categorification

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Categorification

Fruits of Categorification

- Classes of objects (simple, indecomposable projective) give distinguished bases with positivity and integrality properties.
- Uncovers hidden structure in the algebra and its representation.
- Provides tools for studying the category $\mathcal{M}$.
- Applications to topology and physics.

Example

- (Lusztig) Categorification of quantum groups yields canonical bases with positivity and integrality properties.
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Categorification of the Heisenberg Double

Goal

- Find categories whose Grothendieck groups are isomorphic to $\mathfrak{h}$ as $\mathbb{Z}$-modules,
- Find functors lifting the action of $\mathfrak{h}$ on Fock space,
- Find isomorphisms of functors lifting the defining relations of $\mathfrak{h}$.

Module Categories

- $A = \bigoplus_{n \in \mathbb{N}} A_n$: a tower of algebras.
- $A_n$-mod: category of f.g. left $A_n$-modules.
- $A_n$-pmod: category of f.g. projective left $A_n$-modules.
- $G_0(A_n)$: Grothendieck group of $A_n$-mod.
- $K_0(A_n)$: Grothendieck group of $A_n$-pmod.
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Module Categories

- \( A = \bigoplus_{n \in \mathbb{N}} A_n \): a tower of algebras.
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Categorification of the Heisenberg Double

Theorem (Bergeron, Li 2009)

Let \( G(A) = \bigoplus_{n \in \mathbb{N}} G_0(A_n) \) and \( K(A) = \bigoplus_{n \in \mathbb{N}} K_0(A_n) \). Then \((G(A), K(A))\) is a dual pair of Hopf algebras.

Definition (Heisenberg double associated to a tower)

To a tower of algebras \( A \), we associate the Heisenberg double \( h(A) := h(G(A), K(A)) \) and its Fock space \( F(A) = G(A) \).

Theorem (Savage, Yacobi 2015)

The functors \( \text{Ind}_M \) and \( \text{Res}_P \) for \( M \in A\text{-mod} \) and \( P \in A\text{-pmod} \) categorify the Fock space representation \( F(A) = G(A) \).
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**Theorem (Bergeron, Li 2009)**

Let $\mathcal{G}(A) = \bigoplus_{n \in \mathbb{N}} G_0(A_n)$ and $\mathcal{K}(A) = \bigoplus_{n \in \mathbb{N}} K_0(A_n)$. Then $(\mathcal{G}(A), \mathcal{K}(A))$ is a dual pair of Hopf algebras.

**Definition (Heisenberg double associated to a tower)**

To a tower of algebras $A$, we associate the Heisenberg double $\mathcal{H}(A) := \mathcal{H}(\mathcal{G}(A), \mathcal{K}(A))$ and its Fock space $\mathcal{F}(A) = \mathcal{G}(A)$.

**Theorem (Savage, Yacobi 2015)**

The functors $\text{Ind}_M$ and $\text{Res}_P$ for $M \in A\text{-mod}$ and $P \in A\text{-pmod}$ categorify the Fock space representation $\mathcal{F}(A)$ of $\mathcal{H}(A)$. 
Categorification of the Heisenberg Double

**Theorem (Bergeron, Li 2009)**

Let $\mathcal{G}(A) = \bigoplus_{n \in \mathbb{N}} G_0(A_n)$ and $\mathcal{K}(A) = \bigoplus_{n \in \mathbb{N}} K_0(A_n)$. Then $(\mathcal{G}(A), \mathcal{K}(A))$ is a dual pair of Hopf algebras.

**Definition (Heisenberg double associated to a tower)**

To a tower of algebras $A$, we associate the Heisenberg double $\mathfrak{h}(A) := \mathfrak{h}(\mathcal{G}(A), \mathcal{K}(A))$ and its Fock space $\mathcal{F}(A) = \mathcal{G}(A)$.

**Theorem (Savage, Yacobi 2015)**

The functors $\text{Ind}_M$ and $\text{Res}_P$ for $M \in A\text{-mod}$ and $P \in A\text{-pmod}$ categorify the Fock space representation $\mathcal{F}(A)$ of $\mathfrak{h}(A)$. 
Application: $\text{QSym}$ is free over $\text{Sym}$

Tower of 0-Hecke algebras

- $A = \bigoplus_{n \in \mathbb{N}} H_n(0)$
- $G(A) = \text{QSym}, K(A) = \text{NSym}$
- $q = h(\text{QSym}, \text{NSym}):$ quasi-Heisenberg algebra
- $q_{\text{proj}}$: subalgebra generated by $\text{Sym} \subset \text{QSym}$ and $\text{NSym}$.

Theorem (Savage, Yacobi 2015)
Any representation of $q_{\text{proj}}$ generated by a lowest weight vacuum vector is isomorphic to $\text{Sym}$.

Theorem (Hazewinkel 2001, Savage, Yacobi 2015)
$\text{QSym}$ is free as a $\text{Sym}$-module.
Application: \( \text{QSym is free over Sym} \)

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- \( \mathcal{G}(A) = \text{QSym}, \mathcal{K}(A) = \text{NSym} \)
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- \( q_{\text{proj}}: \text{subalgebra generated by Sym} \subset \text{QSym and NSym}. \)

Theorem (Savage, Yacobi 2015)

Any representation of \( q_{\text{proj}} \) generated by a lowest weight vacuum vector is isomorphic to \( \text{Sym} \).

Theorem (Hazewinkel 2001, Savage, Yacobi 2015)

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Further Applications

Towers of Superalgebras

- 0-Hecke-Clifford algebras (Li 2015)
- The ring of peak quasisymmetric functions is free over the subring of symmetric functions spanned by Schur’s $Q$-functions.
- Other towers of (super)algebras (ongoing work)

Thank you for your attention!
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Thank you for your attention!