On $\{3\}$-GDDs with 5 groups

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Definition.

A *group divisible design* ($\{3\}$-GDD) is a decomposition of the complete multipartite graph into triangles called *triples*

- The partite sets are called *groups* or *holes*.

- $\{3\}$-GDD of type $g_1^{t_1}g_2^{t_2} \ldots g_\ell^{t_\ell}$ has $t_i$ groups of size $g_i$, $i = 1, 2, \ldots, \ell$

Ex: a 3-GDD of type $1^33^2$ is a decomposition of $K_{1,1,1,3,3}$ into triangles.

\[
\begin{align*}
F_1 & \rightarrow \{a_1, d_1, e_1\} \\
F_2 & \rightarrow \{a_1, d_2, e_2\} \\
F_3 & \rightarrow \{a_1, d_3, e_3\} \\
\end{align*}
\]

$\{F_1, F_2, F_3\}$ is a one-factorization of $K_{3,3}$ on $\{d_1, d_2, d_3\}$ vs $\{e_1, e_2, e_3\}$

$\{a_1, b_1, c_1\}$ 3-GDD of type $1^3$
Theorem 1. Colbourn 1993 For a 3-GDD of type $g_1g_2 \cdots g_s$ with $g_1 \geq \cdots \geq g_s \geq 1$, $s \geq 2$, and $v = \sum_{i=1}^{s} g_i$ to exist, necessary conditions include:

1. $\left(\begin{array}{c} v \\ 2 \end{array}\right) \equiv \sum_{i=1}^{s} \left(\begin{array}{c} g_i \\ 2 \end{array}\right) \pmod{3}$;

2. $g_i \equiv v \pmod{2}$ for $1 \leq i \leq s$;

3. $g_1 \leq \sum_{i=3}^{s} g_i$;

4. whenever $\alpha_i \in \{0, 1\}$ for $1 \leq i \leq s$ and $v_0 = \sum_{i=1}^{s} \alpha_i g_i$,

$$v_0(v - v_0) \leq 2 \left[ \left(\begin{array}{c} v_0 \\ 2 \end{array}\right) + \left(\begin{array}{c} v - v_0 \\ 2 \end{array}\right) - \sum_{i=1}^{s} \left(\begin{array}{c} g_i \\ 2 \end{array}\right) \right]$$

5. $2g_2g_3 \geq g_1[g_2 + g_3 - \sum_{i=4}^{s} g_i]$; and

6. if $g_1 = \sum_{i=3}^{s} g_i$ then $2g_3g_4 \geq (g_1 - g_2)[g_3 + g_4 - \sum_{i=5}^{s} g_i]$.
What's known

The Colbourn conditions are known to be sufficient when

1. (Wilson 1972) $g_1 = \cdots = g_s$;

2. (Colbourn, Hoffman, and Rees 1992) $g_1 = \cdots = g_{s-1}$ or $g_2 = \cdots = g_s$;

3. (Colbourn, Cusack, and Kreher 1995) $1 \leq t \leq s$, $g_1 = \cdots = g_t$, and $g_{t+1} = \cdots = g_s = 1$;

4. (Bryant and Horsley 2006) $g_3 = \cdots = g_s = 1$; and

5. (Colbourn 1993) $\sum_{i=1}^{s} g_i \leq 60$.

Partial results are known when $g_3 = \cdots = g_s = 2$ (Colbourn, M.A. Oravas, and R. Rees 2000).

Surprisingly, in no other cases are necessary and sufficient conditions known for any other class of 3-gdds (of index 1).
Small number of groups

1. No $3$-GDD with two groups exists;
2. $3$-GDD of type $g_1g_2g_3$ exist if and only if $g_1 = g_2 = g_3$;
3. $3$-GDD with four groups exist if and only if they have type $g^4$ or $g^3u^1$;
4. $3$-GDD with five groups are known to exist of all types $g^5$, $g^4u^1$, and $g_1 \cdots g_5$ with $\sum_{i=1}^{5} g_i \leq 60$, that satisfy the necessary conditions.

However many more cases are possible.

A $3$-GDD with five groups can have all groups of different sizes. A $3$-GDD of type $17^111^19^17^15^1$ exists Colbourn 1993.

The general existence problem for five groups appears to be substantially more complicated than cases with fewer groups.
Type $g^3u^2$

The necessary conditions disencumber for 3-GDDs of type $g^3u^2$. They are simply:

1. $u \equiv 0 \pmod{3}$, and
2. $g \equiv u \pmod{2}$.

We prove that this conditions are also sufficient
Easy result.

Theorem 2. A \( \{3\} \)-GDD of type \( g^3u^2 \) exists whenever \( u = 3g \).
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**Theorem 2.** A $\{3\}$-GDD of type $g^3 u^2$ exists whenever $u = 3g$. 

3-GDD of type $g^3$ (Latin Square)
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One-factorization \( F_1, F_2, \ldots, F_{3g} \) of \( K_{3g,3g} \)

\[ F_x = \{ \{x, a, b\} : \{a, b\} \in F_x \} \]
Rolf Rees established the following:

**Lemma 3.** Let \( h \geq 1 \) and \( 0 \leq r \leq 2h \), \((h, r) \neq (1, 2) \) or \((3, 6)\). There exists a \( \{2, 3\} \)-GDD of type \((2h)^3\) which is resolvable into \( r \) parallel classes of blocks of size 3 and \( 4h - 2r \) parallel classes of blocks of size 2.

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**Theorem 6.** If there exists

then there exist a 3-GDD of type \( g^3 u^2 \).
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**Theorem 6.** If there exists a 3-GDD of type $g^3u^2$ and $g \geq 2$ is such that $g - x$ is even, then there exist a 3-GDD of type $g^3u^2$. Then the following diagram:

```
\begin{align*}
\text{\$g - x$} &
\end{align*}
```

\begin{align*}
\text{\$= 2h$} &
\end{align*}

\begin{align*}
\text{\$u$} &
\end{align*}

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Use the Rees Lemma:
- $r = 2h - u$ parallel classes of triples
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\text{Use the Rees Lemma:} \\
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**Theorem 6.** If there exists

- a 3-GDD of type $x^3u^2$ and
- $g \geq 2x + u$ is such that $g - x$ is even,

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Deconstruct $x$ of the parallel classes of triples into holey one-factors.

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& \text{Deconstruct } x \text{ of the parallel classes of triples into holey one-factors.}
\end{align*}
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\text{a 3-GDD of type } x^3u^2 \text{ and} \\
g \geq 2x + u \text{ is such that } g - x \text{ is even,}
\end{align*}

then there exist a 3-GDD of type \( g^3u^2 \).
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The 31° Theorem.

**Theorem 5.** There exists a \( \{3\}\)-GDD of type \( g^3u^2 \), whenever \( g \geq \frac{5}{3}u \), \( u \equiv 0 \pmod{3} \) and \( g \equiv u \pmod{2} \).
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**Theorem 5.** There exists a $\{3\}$-GDD of type $g^3u^2$, whenever $g \geq \frac{5}{3}u$, $u \equiv 0 \pmod{3}$ and $g \equiv u \pmod{2}$.

Proof.
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**Proof.** We use Theorem 6:

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\begin{cases}
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1. Theorem 4 says: A \( \{3\}\)-GDD of type \( g^3 u^2 \) exists whenever \( u = 3g \).
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2. g \geq 2x + u \\
3. g - x \text{ is even} 
\end{cases} \]

\left\{ \begin{array}{l}
\exists \{3\}\text{-GDD of type } x^3u^2, \\
g \geq 2x + u, \\
g - x \text{ is even}
\end{array} \right\},

\text{then } \exists \{3\}\text{-GDD of type } g^3u^2.

1. Theorem 4 says: A \{3\}-GDD of type $g^3u^2$ exists whenever $u = 3g$.

$\Rightarrow x^3u^2 \text{ exists, where } x = u/3$, because $u \equiv 0 \pmod{3}$. 
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Theorem 5. There exists a $\{3\}$-GDD of type $g^3u^2$, whenever $g \geq \frac{5}{3}u$, $u \equiv 0 \pmod{3}$ and $g \equiv u \pmod{2}$.

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1. \exists \{3\}\text{-GDD of type } x^3u^2 \\
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If the above conditions hold, then \( \exists \{3\}\text{-GDD of type } g^3u^2 \).

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Theorem 5. There exists a $\{3\}$-GDD of type $g^3u^2$, whenever $g \geq \frac{5}{3}u$, $u \equiv 0 \pmod{3}$ and $g \equiv u \pmod{2}$.

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\end{align*}
\]

If \( \exists \{3\}\text{-GDD of type } x^3u^2, \text{ checkmark} \), then \( \exists \{3\}\text{-GDD of type } g^3u^2 \).

1. **Theorem 4 says:** A \( \{3\}\)-GDD of type \( g^3u^2 \) exists whenever \( u = 3g \).

\[ \Rightarrow x^3u^2 \text{ exists, where } x = u/3, \text{ because } u \equiv 0 \pmod{3}. \]

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3. **Modulo 2:** \( g \equiv u = 3x \equiv x \Rightarrow g - x \text{ is even} \).

\( \square \)
Necessary Conditions.

\[ u \leq 3g \]
\[ u \equiv g \pmod{2} \]
\[ u \equiv 0 \pmod{3} \]

Colbourn 1993
Necessary Conditions.

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Theorem 7.

\[ g \geq \frac{5}{3}u \]
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Theorem 7.
\[ g \geq \frac{5}{3} u \]
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Colbourn 1993
\[ 3g + 2u \leq 60 \]
\[ u \equiv g \pmod{2} \]
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Colbourn 1993

\[ 3g + 2u \leq 60 \]
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Giving weight $w$

If a $\{3\}$-GDD of type $g_1g_2g_3g_4g_5$ exists

- replace each point by $w$ points
- build on each triple a $\{3\}$-GDD of type $w^3$
  (for example a Latin Square)

To make a $\{3\}$-GDD of type $(wg_1)(wg_2)(wg_3)(wg_4)(wg_5)$.

**Theorem 7.** If $w$ divides $\gcd(g,u)$, and a $3$-GDD of type $(g/w)^3(u/w)^2$ exists, then a $3$-GDD of type $g^3u^2$ also exists.
3 Mutually Orthogonal Latin Squares

- 3 MOLS of order $k$ is equivalent to a 5-GDD of type $k^5$.
- They are known to exist when $k \neq 2, 3, 6, 10$.
- We give different weights to the points in different groups.

Example:

\[
\begin{array}{cccccc}
  \cdot 3 & \cdot 3 & \cdot 3 & \cdot 3 & \cdot 3 & \cdot 3 \\
  \cdot 3 & \cdot 3 & \cdot 3 & \cdot 3 & \cdot 3 & \cdot 3 \\
  \cdot 3 & \cdot 3 & \cdot 3 & \cdot 3 & \cdot 3 & \cdot 3 \\
  \cdot 3 & \cdot 3 & \cdot 3 & \cdot 9 & \cdot 9 & \cdot 9 \\
\end{array}
\]

We require:
- 3-GDD of type $3^5$
- 3-GDD of type $3^49^1$
- 3-GDD of type $3^39^2$

All < 60 points and satisfy N.C.

Therefore a 3-GDD of type $(15)^3(21)^2$ exists.
Final result

(a) various weights on the points of 5-GDD of type $k^5$s
(b) the $31^\circ$ Theorem, and
(c) Colbourn’s: 3-GDDs on less than 60 points

**Theorem 10.** A 3-GDD of type $g^3u^2$ exist if and only if $u \equiv 0 \pmod{3}$, $u \equiv g \pmod{3}$ except possibly when

\[ g^3u^2 \in \left\{ 9^321^2, 10^324^2, 11^315^2, 11^321^2, 11^327^2, 13^315^2, \\
13^321^2, 13^327^2, 13^333^2, 18^312^2, 18^324^2, 18^342^2, \\
18^348^2, 20^342^2, 20^348^2, 20^354^2, 22^324^2, 22^330^2, \\
22^336^2, 22^342^2, 22^348^2, 22^354^2, 22^360^2, 30^324^2, \\
30^348^2, 30^372^2, 30^384^2, 32^330^2, 32^342^2, 32^354^2, \\
32^366^2, 32^378^2, 32^390^2, 34^324^2, 34^336^2, 34^348^2, \\
34^360^2, 34^372^2, 34^384^2, 34^396^2 \right\} \]
Final result

(a) various weights on the points of 5-GDD of type $k^5$s
(b) the $31^\circ$ Theorem, and
(c) Colbourn’s: 3-GDDs on less than 60 points

**Theorem 10.** A 3-GDD of type $g^3u^2$ exist if and only if $u \equiv 0 \pmod{3}$, $u \equiv g \pmod{3}$ except possibly when

$$g^3u^2 \in \left\{ 9^321^2, 10^324^2, 11^315^2, 11^321^2, 11^327^2, 13^315^2, 13^321^2, 13^327^2, 13^333^2, 18^312^2, 18^324^2, 18^342^2, 18^348^2, 20^342^2, 20^348^2, 20^354^2, 22^324^2, 22^330^2, 22^336^2, 22^342^2, 22^348^2, 22^354^2, 22^360^2, 30^324^2, 30^348^2, 30^372^2, 30^384^2, 32^330^2, 32^342^2, 32^354^2, 32^366^2, 32^378^2, 32^390^2, 34^324^2, 34^336^2, 34^348^2, 34^360^2, 34^372^2, 34^384^2, 34^396^2 \right\}$$

Additional weighing constructions on incomplete group divisible designs found by a Hill climbing algorithm we were able to resolve all of these exceptions.
(a) various weights on the points of 5-GDD of type $k^5$s
(b) the $31^\circ$ Theorem, and
(c) Colbourn’s: 3-GDDs on less than 60 points
(d) Hill climbing
(e) Miscellaneous constructions of 3-I$G$DDs and 3-GDDs.

**Theorem 10.** A 3-GDD of type $g^3u^2$ exist if and only if $u \equiv 0 \pmod{3}$, $u \equiv g \pmod{3}$
(a) various weights on the points of 5-\text{GDD} of type $k^5$s
(b) the 31° Theorem, and
(c) Colbourn’s: 3-\text{GDDS} on less than 60 points
(d) Hill climbing
(e) Miscellaneous constructions of 3-\text{IGDDS} and 3-\text{GDDS}.

\textbf{Theorem 10.} A 3-\text{GDD} of type $g^3u^2$ exist if and only if $u \equiv 0 \pmod{3}$, $u \equiv g \pmod{3}$. 
A general result

An integer partition $g_1 + g_2 + g_3 + g_4 + g_5 = v$ has ratio $(1 : 3)$ if

$$\frac{\min_i g_i}{\max_i g_i} \geq \frac{1}{3}$$

A 3-GDD of type $g_1g_2g_3, g_4, g_5$ has ratio $(1 : 3)$ if its type is a ratio $(1 : 3)$ partition.

**Lemma 11.** If there exits a 5-RDD of type $k^5$ and a ratio $(1 : 3)$ 3-GDD of type $\{r_1, r_2, r_3, r_4, r_5\}$, Then for nonnegative integers $A_i < k$, there exists a ratio $(1 : 3)$ 3-GDD of type $\{g_1, g_2, g_3, g_4, g_5\}$, where

$$g_i = 9k - 6A_i + r_i, \ i = 1, 2, 3, 4, 5.$$  

It follows except for a possible small number of exceptions that ratio $(1 : 3)$ 3-GDDs will exist whenever they satisfy the necessary conditions.