SPHERICAL EMBEDDINGS OF STRONGLY REGULAR GRAPHS

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This is a joint work with Alexander Barg, Kasso Okoudjou, and Wei-Hsuan Yu.
Two-distance tight frames

Spherical two-distance 2-designs

Strongly regular graphs
A finite collection of vectors $S = \{x_i, 1 \leq i \leq N\} \subset \mathbb{R}^n$ is called a finite frame for the Euclidean space $\mathbb{R}^n$ if there are constants $0 < A \leq B < \infty$ such that for all $x \in \mathbb{R}^n$

$$A\|x\|^2 \leq \sum_{i=1}^{N} \langle x, x_i \rangle^2 \leq B\|x\|^2. \tag{1}$$

If $A = B$, then $S$ is called an $A$-tight frame.

An equivalent condition for $A$-tight frames is $Ax = \sum_{i=1}^{N} \langle x, x_i \rangle x_i$ for all $x \in \mathbb{R}^n$.

If in addition $\|x_i\| = 1$ for all $i$, then $S$ is a unit-norm tight frame.
Theorem (Benedetto-Fickus, 2003)

If $N > n$ then

$$\sum_{i,j=1}^{N} \langle x_i, x_j \rangle^2 \geq \frac{N^2}{n}$$

with equality if and only if $S$ is a tight frame.
A finite collection of unit vectors $S \subseteq \mathbb{R}^n$ is called a spherical two-distance set if there are two numbers $a$ and $b$ such that the inner products of distinct vectors from $S$ are either $a$ or $b$. If at the same time $S$ is a finite unit-norm tight frame, we call it a two-distance tight frame.

If $a + b \neq 0$, the definition of a tight frame immediately shows that $S$ must be regular, i.e. the distribution of inner products is the same for each vector $x_i$. 
If the two inner products of a two-distance tight frame $S$ satisfy the condition $a = -b$, then it is called an equiangular tight frame.

**Equiangular tight frames**

**Certain strongly regular graphs**

For a natural number $t$, a finite set of vectors $S = \{x_i, 1 \leq i \leq N\} \subset S^{n-1}$ is called a spherical $t$-design if for any polynomial $f(x)$ of degree at most $t$

$$\frac{1}{|S^{n-1}|} \int_{x \in S^{n-1}} f(x) d\sigma(x) = \frac{1}{N} \sum_{i=1}^{n} f(x_i).$$

(3)

Examples:

- Icosahedron and dodecahedron are 5-designs
- 120-cell and 600-cell are 11-designs
- Root systems
- Minimal vectors of the Leech lattice form an 11-design
Spherical 2-designs are tight frames

$S = \{x_i, 1 \leq i \leq N\} \subset \mathbb{S}^{n-1}$ is a spherical 2-design if and only if

$$\sum_{i,j=1}^{N} \langle x_i, x_j \rangle^2 = \frac{N^2}{n} \quad \text{and} \quad \sum_{i=1}^{N} x_i = 0$$

(4)
Spherical 2-designs are tight frames

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(4)
A regular graph of degree \( k \) on \( v \) vertices is called strongly regular if every two adjacent vertices have \( \lambda \) common neighbors and every two non-adjacent vertices have \( \mu \) common neighbors. We use the notation \( \text{SRG}(v, k, \lambda, \mu) \) to denote such a graph.

Examples:

- Cycle of length 5
- Petersen graph
- Hoffman-Singleton graph
- Conference graphs
- \( n \times n \) rook’s graphs
Delsarte, Goethals, and Seidel obtained a spherical embedding of $\Gamma = \text{SRG}(v, k, \lambda, \mu)$ by associating a basis of $\mathbb{R}^v$ with the vertices of $\Gamma$, projecting these vectors on an eigenspace of the adjacency matrix of $\Gamma$, and normalizing lengths of projections. They also showed that this embedding forms a two-distance 2-design.
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Proposition

If $S$ is a regular 2-distance tight frame in $\mathbb{R}^n$, then $S$ is either an $n$-dimensional spherical 2-design, or is similar to an $(n - 1)$-dimensional spherical 2-design contained in a subsphere of radius $\sqrt{1 - 1/n}$.

Proof.

Let $s = \sum_{i=1}^{N} x_i$. The value $t := \langle x_i, s \rangle$ is the same for all $i$. Using an equivalent definition of tight frames, we get

$$\frac{N}{n} s = \sum_{i=1}^{N} tx_i = ts.$$ 

Hence either $s = 0$ or $t = \frac{N}{n}$. \qed
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If $S$ is a regular 2-distance tight frame in $\mathbb{R}^n$, then $S$ is either an $n$-dimensional spherical 2-design, or is similar to an $(n - 1)$-dimensional spherical 2-design contained in a subsphere of radius $\sqrt{1 - 1/n}$. 

Regular two-distance tight frames

Spherical two-distance 2-designs
Proposition

If \( S \) is a regular two-distance tight frame, then its associated graph \( \Gamma_1 \) (and \( \Gamma_2 \) as the complement of \( \Gamma_1 \)) is a strongly regular graph.

Proof.

Use a theorem by Delsarte, Goethals, Seidel for 2-designs or just check the definition of tight frames carefully.
Proposition

If $S$ is a regular two-distance tight frame, then its associated graph $\Gamma_1$ (and $\Gamma_2$ as the complement of $\Gamma_1$) is a strongly regular graph.
TWO-DISTANCE TIGHT FRAMES ARE DEFINED BY SRG’S

Spherical two-distance 2-designs

Strongly regular graphs

**Question**

What two-distance spherical embeddings of SRG’s form 2-designs?
For a given SRG($v, k, \lambda, \mu$) which is not a complete or empty graph, its adjacency matrix has three mutually orthogonal eigenspaces (subspaces) that correspond to three eigenvalues: the all-one vector $\mathbf{1}$ with eigenvalue $k$ and subspaces $E_1$ and $E_2$. Projecting an orthonormal basis of $\mathbb{R}^n$ on $\mathbf{1}$ and normalizing gives a trivial 1-dimensional embedding, where all inner products are 1.

Projections on $E_1$ or on $E_2$ after normalization give two-distance 2-designs.

Direct orthogonal sum of two spherical embeddings is a spherical embedding.
Proposition

For a given $\Gamma = \text{SRG}(N, k, \lambda, \mu)$, any two-distance spherical embedding may be represented as a direct orthogonal sum of the trivial and Delsarte-Goethals-Seidel embeddings.

Proof.

Since the Gram matrix is positive definite, the set of possible values of scalar products $a$ and $b$ associated to embeddings of $\Gamma$ forms a triangle on $(a, b)$-plane with vertices corresponding to the trivial and two Delsarte-Goethals-Seidel embeddings. Therefore, any pair $(a, b)$ may be obtained as a non-negative linear combination of scalar products from these embeddings.
Theorem

Any spherical two-distance 2-design with graph \( \Gamma = \text{SRG}(N, k, \lambda, \mu) \) for one of the distances is either one of two Delsarte-Goethals-Seidel embeddings, or a regular \((N - 1)\)-dimensional simplex.

Proof.

Use the previous proposition and the description of embeddings via eigenspaces of the adjacency matrix of \( \Gamma \). \[\square\]
Theorem

Any spherical two-distance 2-design with graph \( \Gamma = \text{SRG}(N, k, \lambda, \mu) \) for one of the distances is either one of two Delsarte-Goethals-Seidel embeddings, or a regular \((N - 1)\)-dimensional simplex.
Theorem

Let $S$ be a regular two-distance tight frame in $\mathbb{R}^n$. Then $S$ forms a spherical two-distance 2-design or a shifted 2-design. In either case $S$ can be obtained as a spherical embedding of a strongly regular graph. Under spherical embedding, every strongly regular graph gives rise to three different two-distance 2-designs and therefore, to six different two-distance tight frames, two of which are regular simplices.
<table>
<thead>
<tr>
<th>SRG($N, k, \lambda, \mu$)</th>
<th>2-design ($n, N, a, b$)</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>shifted 2-design ($n, N, a, b$)</td>
</tr>
<tr>
<td>$(10, 6, 3, 4)$</td>
<td>$(4, 10, \frac{1}{6}, -\frac{2}{3}); (5, 10, \frac{1}{3}, -\frac{1}{3});$</td>
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<td>$(5, 10, \frac{1}{3}, -\frac{1}{3}); (6, 10, \frac{4}{9}, -\frac{1}{9})$</td>
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<tr>
<td>$(15, 8, 4, 4)$</td>
<td>$(5, 15, \frac{1}{4}, -\frac{1}{2}); (9, 15, \frac{1}{6}, -\frac{1}{4});$</td>
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<td>$(6, 15, \frac{3}{8}, -\frac{1}{4}); (10, 15, \frac{1}{4}, -\frac{1}{8})$</td>
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<tr>
<td>$(16, 10, 6, 6)$</td>
<td>$(5, 16, \frac{1}{5}, -\frac{3}{5}); (10, 16, \frac{1}{5}, -\frac{1}{5});$</td>
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<td>$(6, 16, \frac{1}{3}, -\frac{1}{3}); (11, 16, \frac{3}{11}, -\frac{1}{11})$</td>
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THANK YOU!