Oct 16, 2009

MA3160-04 Exam 1

No calculators! Justify all answers! **Name**(print): **Solutions**

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1. **(10 pts.)** Find the equation of the linear function $z = mx + ny + c$ whose graph contains the points $P_1 = (1, 0, 0)$, $P_2 = (0, -1, 1)$, $P_3 = (0, 0, 2)$.

Solution

The three points must satisfy the equation $z = mx + ny + c$

$$\begin{cases} P_1 = (1,0,0): & 0 = m + 0 + c \\ P_2 = (0,-1,1): & 1 = 0 - n + c \\ P_3 = (0,0,2): & 2 = 0 + 0 + c \end{cases} \implies \begin{cases} -m &= c \\ 1 + n &= c \\ c &= 2 \end{cases} \implies \begin{cases} \mathbf{m = -2} \\ \mathbf{n = 1} \\ \mathbf{c = 2} \end{cases}$$

Therefore, the equation of the linear function is

$$\boxed{z = -2x + y + 2}$$

2. Let $f(x, y) = \sqrt{x^2 + y^3}$.

(a) **(10 pts.)** Find the differential of $f(x, y)$ at the point (1,2).

Solution

Note that

$$f_x(x, y) = \frac{x}{\sqrt{x^2 + y^3}} \quad \text{and} \quad f_y(x, y) = \frac{3y^2}{2\sqrt{x^2 + y^3}}$$

Therefore,

$$df = f_x(x, y)dx + f_y(x, y)dy = \frac{x}{\sqrt{x^2 + y^3}}dx + \frac{3y^2}{2\sqrt{x^2 + y^3}}dy$$

and at the point (1,2)

$$\boxed{df = \frac{1}{3}dx + 2dy}$$

(b) **(5 pts.)** Use part (a) to estimate $f(1.1, 1.9)$.

Solution

$$f(1.1, 1.9) \approx f(1, 2) + f_x(1, 2)\Delta x + f_y(1, 2)\Delta y = 3 + \frac{1}{3}(0.1) + 2(-0.1) = 3 + \frac{1}{30} - \frac{1}{5}$$

Therefore,

$$\boxed{f(1.1, 1.9) \approx 2\frac{5}{6} \approx 2.833}$$

3. **(12 pts.)** Find the directional derivative of $f(x, y, z) = x^2 + xy - z^2$ at the point $P = (1, 1, 0)$ in the direction of the point $Q = (1, 0, 1)$.

Solution

The direction vector is

$$\vec{v} = \vec{PQ} = (1-1)\vec{i} + (0-1)\vec{j} + (1-0)\vec{k} = -\vec{j} + \vec{k}$$

with length

$$\|\vec{v}\| = \sqrt{0^2 + (-1)^2 + 1^2} = \sqrt{2}.$$

Therefore, the *unit* direction vector is

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{-1}{\sqrt{2}}\vec{j} + \frac{1}{\sqrt{2}}\vec{k}.$$

The gradient of $f(x, y, z)$ at $P = (1, 1, 0)$ is

$$\textbf{grad}\, f(1,1,0) = f_x(1,1,0)\vec{i} + f_y(1,1,0)\vec{j} + f_z(1,1,0)\vec{k} = ((2x+y)\vec{i} + x\vec{j} - 2z\vec{k})|_{(1,1,0)} = 3\vec{i} + \vec{j}.$$

Therefore, the derivative in the direction $\vec{u}$ at the point $P = (1, 1, 0)$ is

$$\boxed{f_{\vec{u}}(1,1,0) = \textbf{grad}\, f(1,1,0) \cdot \vec{u} = 0 + \frac{-1}{\sqrt{2}} + 0 = \frac{-1}{\sqrt{2}}}$$

4. **(13 pts.)** Find the equation of the tangent plane at $P = (1, -1, 1)$ to the surface $x^2 + y^2 - xyz = 3$

Solution

Note that the point $P = (1, -1, 1)$ actually lies on the surface $x^2 + y^2 - xyz = 3$.
The surface $x^2 + y^2 - xyz = 3$ is a level surface of the function $w = f(x, y, z) = x^2 + y^2 - xyz$.
Therefore, the normal vector at $P = (1, -1, 1)$ to the surface $x^2 + y^2 - xyz = 3$ is

$$\text{grad}\, f(1,-1,1) = f_x(1,-1,1)\vec{i} + f_y(1,-1,1)\vec{j} + f_z(1,-1,1)\vec{k} = [(2x-yz)\vec{i} + (2y-xz)\vec{j} - xy\vec{k}]|_{(1,-1,1)} = 3\vec{i} - 3\vec{j} + \vec{k}.$$

Let $Q = (x, y, z)$ be a general point on the tangent plane. Then the vector

$$\vec{v} = \vec{PQ} = (x-1)\vec{i} + (y+1)\vec{j} + (z-1)\vec{k}$$

lies on the tangent plane and is perpendicular to grad $f(1, -1, 1)$.
Therefore, grad $f(1, -1, 1) \cdot \vec{v} = 0$ gives the equation of the tangent plane, i.e.,

$$\boxed{3(x-1) - 3(y+1) + (z-1) = 0}$$

5. (10 pts.) The air pressure is dropping at a constant rate with respesctc to time everywhere. In the eastward direction, the air pressure decreases at a rate of 2 pascal per kilometer. A ship sailing eastward past an island records a pressure drop of 50 pascals in 2 hours. The time rate decrease of the air pressure on the island is 3 pascals per hour. Estimate the (average) speed of the ship.

Solution

The time rate of change of the air pressure on the island (at fixed x) is $\dfrac{\partial p(x,t)}{\partial t} = -3$ Pa/hr, and the pressure drop in the eastward direction (at constant t) is $\dfrac{\partial p(x,t)}{\partial x} = -2$ Pa/km. The pressure drop measured on the ship is the time change for varying x and varying t, that is, $\dfrac{\mathrm{d}p(x,t)}{\mathrm{d}t} = -50/2 = -25$ Pa/hr. We want the speed of the ship which is $\dfrac{\partial x}{\partial t}$. Therefore, by the chain rule, $\dfrac{\mathrm{d}p(x,t)}{\mathrm{d}t} = \dfrac{\partial p}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial p}{\partial t}$ and, therefore, solving for $\dfrac{\partial x}{\partial t}$ gives

$$\boxed{\frac{\partial x}{\partial t} = \left(\frac{\mathrm{d}p(x,t)}{\mathrm{d}t} - \frac{\partial p}{\partial t}\right)\Big/\frac{\partial p(x,t)}{\partial x} = \frac{-25-(-3)}{-2} = 11 \text{ km/hr.}}$$

6. (15 pts.) Let $f(x,y) = \dfrac{1}{xy}$. Find the quadratic Taylor polynomial $Q(x,y)$ valid near (2,1).

Solution

$$Q(x,y) = f(2,1)+f_x(2,1)(x-2)+f_y(2,1)(y-1)+\frac{1}{2}f_{xx}(2,1)(x-2)^2+f_{xy}(2,1)(x-2)(y-1)+\frac{1}{2}f_{yy}(2,1)(y-1)^2$$

$$\begin{aligned} f(2,1) &= \frac{1}{2} & f_{xx}(2,1) &= \left.\frac{2}{x^3y}\right|_{(2,1)} = \frac{1}{4} \\ f_x(2,1) &= \left.\frac{-1}{x^2y}\right|_{(2,1)} = -\frac{1}{4} & f_{xy}(2,1) &= \left.\frac{1}{x^2y^2}\right|_{(2,1)} = \frac{1}{4} \\ f_y(2,1) &= \left.\frac{-1}{xy^2}\right|_{(2,1)} = -\frac{1}{2} & f_{yy}(2,1) &= \left.\frac{2}{xy^3}\right|_{(2,1)} = 1 \end{aligned}$$

Therefore, the quadratic Taylor polynomial is

$$\boxed{Q(x,y) = \frac{1}{2} - \frac{1}{4}(x-2) - \frac{1}{2}(y-1) + \frac{1}{8}(x-2)^2 + \frac{1}{4}(x-2)(y-1) + \frac{1}{2}(y-1)^2}$$

7. (12 pts.) Use Lagrange multipliers to find the maximum and minimum values of $f(x, y) = x + 4y$ subject to the constraint $x^2 + y^2 = 17$.

Solution

The objective function is $f(x, y) = x + 4y$ and the constraint is $g(x, y) = x^2 + y^2 = 17$. The Lagrange multiplier method requires that grad $f(x, y) = \lambda$grad $g(x, y)$, which leads to the system of equations

$$\begin{cases} f_x(x,y) &= \lambda g_x(x,y) \\ f_y(x,y) &= \lambda g_y(x,y) \end{cases} \Longrightarrow \begin{cases} 1 &= 2\lambda x \\ 4 &= 2\lambda y \end{cases} \Longrightarrow \begin{cases} \lambda &= 1/(2x) \\ \lambda &= 2/y \end{cases} \Longrightarrow y = 4x.$$

Substitution of $y = 4x$ into the constraint equation yields

$$x^2 + (4x)^2 = 17 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1.$$

Consequently, $y = \pm 4$ and the critical points are $\boxed{\{(1, 4), (-1, -4)\}.}$

Evaluation of the objective function at the critical point gives $f(1, 4) = 17$ and $f(-1, -4) = -17$, which shows that there is a

$$\boxed{\text{maximum at } (1,4) \text{ and a minimum at } (-1,-4)}$$

8. (13 pts.) Find the point(s) on the surface $z^2 - xy = 1$ that is (are) closest to the origin.

Solution

The square of the distance of a point $P = (x, y, z)$ to the origin is $s(x, y, z) = x^2 + y^2 + z^2$. Substitution of $z^2 = 1 + xy$ into $s(x, y, z)$ yields

$$f(x, y) = 1 + xy + x^2 + y^2.$$

The critical points of f(x,y) are obtained from grad $f(x, y) = 0$. This leads to

$$\begin{cases} f_x(x,y) &= y + 2x &= 0 \\ f_y(x,y) &= x + 2y &= 0 \end{cases} \Longrightarrow y = -2x \Longrightarrow x + 2(-2x) = 0 \Longrightarrow \begin{cases} x &= 0 \\ y &= 0 \\ z &= \pm 1 \end{cases}$$

Consequently, the two critical points are $\boxed{\{(0, 0, 1), (0, 0, -1)\}.}$

Second derivative test: $D(x, y) = f_{xx} f_{yy} - f_{xy}^2 = 2 \cdot 2 - (1)^2 = 3 > 0$ and $f_{xx} = 2 > 0$.

This implies that **both critical points are local minima.**

Check for global minima:

Because $f(x, y) = 1 + xy + x^2 + y^2 \to \infty$ as $x, y \to \infty$, **both critical points are also global minima.**

or, by completing the square, $f(x, y) = 1 + x^2 + xy + y^2/2 - y^2/2 + y^2 = 1 + (x + y/2)^2 + y^2/2 \geq 1$ for all points away from the origin, and, therefore, **both critical points are also global minima.**