Section 5.2

#13.Let N(t) be the number of wrong calls with in t days. $\{N(t) : t \geq 0\}$ is a Poisson process with $\lambda = 1/7$. By the independent property and stationary, the desired probability is

$$P(N(1) = 0) = e^{-(1/7)\times 1} = 0.87.$$

#14. Choose one month as unit of time. When $\lambda = 5$ and the probability of no crime during any given month of year is

$$P(N(1) = 0) = e^{-5} = 0.0067.$$

Hence the desired probability is

$$\binom{12}{2}(0.0067)^2(1 - 0.0067) = 0.0028.$$

#15. Choose one day as the time unit of time. Then $\lambda = 3$ and the probability of no accident in one day is

$$P(N(1) = 0) = 0.0498$$

The number of days without any accidents in January is approximately another Poisson random variable with approximate rate $31 \times 0.05 = 1.55$. Hence, the desired probability is

$$\frac{(1.55)^3}{3!}e^{-1.55} = 0.13.$$

Section 6.1

#1. (a) $1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^\infty ce^{-3x}dx = c/3$, we get $c = 3$.

(b) $P(0 < X < 1/2) = \int_0^{0.5} 3e^{-3x}dx = 1 - e^{-3/2} = 0.78$.

#5. (a) Using the property $\int_{-\infty}^{\infty} f(x)dx = 1$, it follows

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f(x)dx \\ &= c\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}dx \\ &= c \arcsin(x)|_{-1}^{1} = c\pi. \end{aligned}$$

So, $c = 1/\pi$.

(b) if $x < -1$, $F(x) = 0$; if $x > 1$, $F(x) = 1$; if $-1 < x < 1$,

$$F(x) = \int_{-1}^{x} \frac{1}{\pi\sqrt{1-t^2}} dt = \frac{1}{\pi}(\arcsin t|_{-1}^{x}) = \frac{1}{\pi}\arcsin x - \frac{1}{2}.$$

Thus,

$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{\pi}\arcsin x - \frac{1}{2} & -1 < x < 1 \\ 1 & x \geq 1. \end{cases}$$

#6. A function $h(x)$ is a density function if the function $h(x)$ satisfy two conditions (1) non-negative and (2) $\int_{-\infty}^{\infty} h(x)dx = 1$. It is obvious that $h(x)$ satisfy the first condition. We need to prove that $h(x)$ satisfy the second condition.

$$\begin{aligned} \int_{-\infty}^{\infty} h(x)dx &= \int_{\alpha}^{\infty} h(x)dx \\ &= \frac{1}{1-F(\alpha)} \int_{\alpha}^{\infty} f(x)dx \\ &= \frac{1}{1-F(\alpha)} P(X > \alpha) \\ &= \frac{1}{1-F(\alpha)} (1 - F(\alpha)) \\ &= 1. \end{aligned}$$

#9. You can consider each radio is a Bernoulli trial with "success"="last more than 15 year", and

$$\begin{aligned} p &= P(success) = P(X > 15) \\ &= \int_{15}^{\infty} \frac{1}{15} e^{-x/15} dx = \frac{1}{e}. \end{aligned}$$

Thus the answer is

$$\sum_{i=4}^{8} \begin{pmatrix} 8 \\ i \end{pmatrix} \left(\frac{1}{e}\right)^{i} \left(1 - \frac{1}{e}\right)^{8-i} = 0.3327.$$

Section 6.2

#1. Let G be the distribution of Y,

$$G(y) = P(Y \leq y) = P(X^3 \leq y) = P(X \leq \sqrt[3]{y})$$

It is easy to see that if $\sqrt[3]{y} < -2$ i.e. $y < -8$, $G(y) = 0$; if $\sqrt[3]{y} > 2$ i.e. $y > 8$, $G(y) = 1$; For $-2 \leq \sqrt[3]{y} \leq 2$ i.e. $-8 \leq y \leq 8$,

$$\begin{aligned} G(y) &= P(x \leq \sqrt[3]{y}) \\ &= \int_{-2}^{\sqrt[3]{y}} \frac{1}{4} dx = \frac{1}{4}\sqrt[3]{y} + \frac{1}{2}. \end{aligned}$$

Therefore,

$$G(y) = \begin{cases} 0 & y < -8 \\ \frac{1}{4}\sqrt[3]{y} + \frac{1}{2} & -8 \leq y \leq 8 \\ 1 & y > 8. \end{cases}.$$

This gives,

$$g(y) = G'(y) = \begin{cases} \frac{1}{12} y^{-2/3} & -8 < y < 8 \\ 0 & \text{otherwise} \end{cases}.$$

Let H be the distribution function of Z. Then

$$\begin{aligned} H(z) &= P(Z \leq z) \\ &= P(X^4 \leq z) \end{aligned}$$

if $z < 0$, $H(z) = 0$; if $z \geq 0$;

$$\begin{aligned} H(z) &= P(X^4 \leq z) \\ &= P(-\sqrt[4]{z} \leq x \leq \sqrt[4]{z}) \\ &= \int_{-\sqrt[4]{z}}^{\sqrt[4]{z}} f(x) dx \\ &= \begin{cases} \int_{-\sqrt[4]{z}}^{\sqrt[4]{z}} \frac{1}{4} dx = \frac{1}{2}\sqrt[4]{z} & 0 \leq \sqrt[4]{z} < 2 \text{ or } 0 \leq z < 16 \\ 1 & \sqrt[4]{z} \geq 2 \text{ or } z \geq 16. \end{cases} \end{aligned}$$

In summary

$$H(z) = \begin{cases} 0 & z < 0 \\ \frac{1}{2}\sqrt[4]{z} & 0 \le z < 16 \\ 1 & z \ge 16. \end{cases}.$$

This gives,

$$h(z) = H'(z) = \begin{cases} \frac{1}{8}z^{-3/4} & 0 < y < 16 \\ 0 & \text{otherwise} \end{cases}.$$

#3. The set of possible value of X is A=(0,∞). Let $h(x) = x^{3/2}$; The set of possible value of $h(x) = (0, \infty)$. The inverse of h is g, where $g(y) = y^{2/3}$. Thus $g'(y) = 2/(3y^{1/3})$ and hence

$$f_Y(y) = \frac{2}{3y^{1/3}}e^{-y^{2/3}},\ y \in (0, \infty).$$

To find the pdf of $Z = e^{-X}$, let $h(x) = e^{-x}$. The possible value of $h(x)$ is B=(0,1). the inverse function of h is $g(z) = -\ln z$. So, $g'(z) = -\frac{1}{z}$. Therefore,

$$f_Z(z) = e^{-(-\ln z)}|-\frac{1}{z}| = 1,\ z \in (0, 1);$$

0, otherwise.

#5. Let G and g be the distribution function and the pdf of Y, respectively. Then

$$\begin{aligned} G(y) &= P(Y \le y) \\ &= P(\sqrt[3]{X^2} \le y) \end{aligned}$$

If $y < 0$, then $G(y) = 0$; if $y \ge 0$,

$$\begin{aligned} G(y) &= P(\sqrt[3]{X^2} \le y) \\ &= P(-y^{3/2} \le X \le y^{3/2}) \\ &= \int_{-y^{3/2}}^{y^{3/2}} f(x)dx \\ &= \int_0^{y^{3/2}} \lambda e^{-\lambda x}dx \\ &= 1 - e^{-\lambda y^{3/2}} \end{aligned}$$

S0,

$$g(y) = G'(y) = \frac{3\lambda}{2} y^{1/2} e^{-\lambda y^{3/2}}, \ y \geq 0;$$

0, otherwise.