• Solution of Section 8.1 and 8.2.

Section 8.1

1.(a).

$$\sum_{x=1}^{2}\sum_{y=1}^{2} k(x/y) = 1$$

imply that $k = 2/9/$

(b)

$$p_X(x) = \sum_{y=1}^{2} \frac{2x}{9y} = \frac{x}{3},\ x = 1, 2$$

$$p_Y(y) = \sum_{x=1}^{2} \frac{2x}{9y} = \frac{2}{3y},\ y = 1, 2$$

(c).

$$P(X > 1|Y = 1) = P(X = 2|Y = 1) = \frac{p(2,1)}{p_Y(1)} = \frac{4/9}{2/3} = \frac{2}{3}.$$

3. (a) $k(1+1+1+9+4+9) = 1$ imply that $k = 1/25$.

(b)The marginal probability function of X is given by

$$\begin{aligned} p_X(1) &= p(1,1) + p(1,3) = 12/25 \\ p_X(2) &= p(2,3) = 13/25 \end{aligned}$$

The marginal probability function of Y is given by

$$\begin{aligned} p_Y(1) &= p(1,1) = 2/25 \\ p_Y(3) &= p(1,3) + p(2,3) = 23/25. \end{aligned}$$

9. (a)

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\ &= \int_{0}^{x} 2dy = 2x,\ 0 \le x \le 1 \end{aligned}$$

$$\begin{aligned} f_y(x) &= \int_{-\infty}^{\infty} f(x,y)dx \\ &= \int_y^1 2dx = 2(1-y),\ 0 \le y \le 1 \end{aligned}$$

(b).

$$P(X < 0.5) = \int_0^{0.5} f_x(x)dx = \int_0^{0.5} 2xdx = \frac{1}{4}$$

$$\begin{aligned} P(X &< 2Y) = \int\int_{x<2y} f(x,y)dxdy \\ &= \int_0^1 \int_{x/2}^x 2dydx = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} P(X &= Y) = \int\int_{x=y} f(x,y)dxdy \\ &= \int_0^1 \int_x^x 2dydx = 0. \end{aligned}$$

11.

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\ &= \int_0^2 \frac{1}{2}ye^{-x}dy = e^{-x}, x > 0. \end{aligned}$$

$$f_Y(y) = \int_0^{\infty} \frac{1}{2}ye^{-x}dx = \frac{1}{2}y,\ 0 < y < 2.$$

13. The area of R is $\int_0^1 (x-x^2)dx = \frac{1}{6}$, so

$$f(x,y) = \begin{cases} 6 & \text{if } (x,y) \in R \\ 0 & \text{elsewhere.} \end{cases}$$

16. The problem is equivalent to the following: Two random variable X and Y are selected at random and independently from $(0,l)$. What ia the

probability that $|X-Y|<X$? The joint pdf of X and Y is given

$$f(x,y)=\begin{cases}\frac{1}{l^2} & 0<x<l, 0<y<l.\\ 0 & \text{elsewhere.}\end{cases}$$

So,

$$\begin{aligned}P(|X-Y| &< X)=P(-X<X-Y<X)\\ &= P(2X>Y)\\ &= \int_0^l\int_{y/2}^l \frac{1}{l^2}dxdy=\frac{3}{4}.\end{aligned}$$

Section 8.2

1. Note that

$$\begin{aligned}p_X(x) &= \frac{1}{25}(3x^2+5)\\ p_Y(y) &= \frac{1}{25}(2y^2+5).\end{aligned}$$

Now $p_X(1)=\frac{8}{25}$, $p_Y(0)=\frac{5}{25}$ and $p(1,0)=\frac{1}{25}$. Since $p(1,0)\neq p_X(1)p_Y(0)$, X and Y are dependent.

3. By the independent of X and Y,

$$\begin{aligned}P(X &= 1,Y=3)=P(X=1)P(Y=3)\\ &= \frac{1}{2}(\frac{2}{3})\frac{1}{2}(\frac{2}{3})^3=\frac{4}{81}.\end{aligned}$$

$$\begin{aligned}P(X+Y &= 3)=P(X=1,Y=2)+P(X=2,Y=1)\\ &= P(X=1)P(Y=2)+P(X=2)P(Y=1)\\ &= \frac{1}{2}(\frac{2}{3})\frac{1}{2}(\frac{2}{3})^2+\frac{1}{2}(\frac{2}{3})^2\frac{1}{2}(\frac{2}{3})=\frac{4}{27}\end{aligned}$$

8. For $i,j\in\{0,1,2,3\}$, the sum of the numbers in the ith row is $p_X(i)$ and the sum of the numbers in the jth row is $p_Y(i)$. We have that

$$p_X(0)=0.41,\ p_X(1)=0.44,\ p_X(2)=0.14,\ p_X(3)=0.01,$$

$$p_y(0) = 0.41,\ p_Y(1) = 0.44,\ p_Y(2) = 0.14,\ p_Y(3) = 0.01.$$

Since for all $x, y \in \{0, 1, 2, 3\}, p(x, y) = p_X(x)p_Y(Y)$, X and Y are independent.

9. There are no independent because

$$\begin{aligned} f_X(x) &= \int_0^x 2dy = 2x, 0 \leq x \leq 1; \\ f_Y(y) &= \int_y^1 2dx = 2(1-y), 0 \leq y \leq 1; \end{aligned}$$

and so $f(x, y) \neq f_X(x)f_Y(y)$.

• Solution of Section 9.1 to 9.3.

Section 9.1.

#1. Since

$$E(X) = \int_0^1 x(1-x)dx + \int_1^2 x(x-1)dx = 1$$

and

$$E(X^2) = \int_0^1 x^2(1-x)dx + \int_1^2 x^2(x-1)dx = 1.5,$$

we have $E(X^2 + X) = 1.5 + 1 = 2.5$.

3. We have that

$$E(X^2) = Var(X) + (EX)^2 = 1.$$

Similarly, $E(Y^2) = E(Z^2) = 1$. Thus.

$$\begin{aligned} E[X^2(Y+5Z)^2] &= E(X^2)E[(Y+5Z)^2] \\ &= E[Y^2 + 10YZ + 25Z^2] \\ &= E(Y^2) + 25E(Z^2) + 10E(Y)E(Z) \\ &= 26. \end{aligned}$$

$\#$ 4. Since $f(x,y) = e^{-x} \times 2e^{-2y}$, $x > 0, y > 0$. we can get that X and Y are independent exponential random variables with parameters 1 and 2. So,

$$\begin{aligned} E(X) &= 1,\ Var(X) = 1 \\ E(Y) &= \frac{1}{2},\ Var(Y) = \frac{1}{4}. \end{aligned}$$

and

$$\begin{aligned} E(X^2) &= Var(X) + (EX)^2 = 2 \\ E(Y^2) &= Var(Y) + (EY)^2 = \frac{1}{2} \end{aligned}$$

So, $E(X^2 + Y^2) = E(X^2) + E(Y^2) = 2.5$.

Section 9.2

$\#$ 1. (a) and (b) are similar, we only give the proof of (a) here.

$$\begin{aligned} Cov(X+Y, Z) &= E[(X + Y - E(X+Y))(Z - EZ)] \\ &= E[(X - EX + Y - EY)(Z - EZ)] \\ &= E[(X - EX)(Z - EZ) + (Y - EY)(Z - EZ)] \\ &= E[(X - EX)(Y - EY)] + E[(Y - EY)(Z - EZ)] \\ &= Cov(X, Z) + Cov(Y, Z). \end{aligned}$$