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Chapter 1

Classification of Differential Equations

1. (a) First-order, homogeneous, linear, nonconstant-coefficient, scalar ODE.
   (b) Second-order, inhomogeneous, linear, constant-coefficient, scalar PDE.
   (c) First-order, nonlinear, scalar PDE.

3. (a) Second-order, nonlinear, scalar ODE.
   (b) First-order, inhomogeneous, linear, constant-coefficient, scalar PDE.
   (c) Second-order, inhomogeneous, linear, nonconstant-coefficient, scalar PDE.

5. (a) No.
   (b) Yes.
   (c) No.

7. There is only one such $f$: $f(t) = t \cos(t)$.

9. For any constant $C \neq 1$, the function $w(t) = C' u(t)$ is a different solution of the differential equation.
Chapter 2

Models in one dimension

2.1 Heat Flow in a bar; Fourier’s law

1. The units of $\kappa$ are energy per length per time per temperature, for example, $J/(\text{cm s K}) = W/(\text{cm K})$.

3. The units of $A\rho c\Delta x$ are $\text{cm}^2 \text{ g} / \text{cm}^3 \text{ g K} = J$.

5. $\frac{du}{dt}(\ell, t) = \frac{r}{\kappa}$, $t > t_0$.

7. Measuring $x$ in centimeters, the steady-state temperature distribution is $u(x) = 0.1x + 20$, the solution of

$$-\frac{d^2 u}{dx^2} = 0, \ u(0) = 20, \ u(100) = 30.$$ 

The heat flux is

$$-\kappa \frac{du}{dx}(x) = -0.802 \cdot 0.1 = -0.0802 \text{ W/cm}^2.$$

Since the area of the bar’s cross-section is $\pi \text{ cm}^2$, the rate at which energy is flowing through the bar is $0.0802\pi \approx 0.252 \text{ W}$ (in the negative direction).

9. Consider the right endpoint ($x = \ell$). Let the temperature of the bath be $u_\ell$, so that the difference in temperature between the bath and the end of the bar is $u(\ell, t) - u_\ell$. The heat flux at $x = \ell$ is, according to Fourier’s law,

$$-\kappa \frac{du}{dx}(\ell, t),$$

so the statement that the heat flux is proportional to the temperature flow is written

$$-\kappa \frac{du}{dx}(\ell, t) = \alpha (u(\ell, t) - u_\ell),$$

where $\alpha > 0$ is a constant of proportionality. This can be simplified to

$$\alpha u(\ell, t) + \kappa \frac{du}{dx}(\ell, t) = \alpha u_\ell.$$

The condition at the right end is similar:

$$\alpha u(0, t) - \kappa \frac{du}{dx}(0, t) = \alpha u_0.$$

(The sign change appears because, at the left end, a positive heat flux means that heat flows into the bar, while at the right end the opposite is true.)

11. The IBVP is

$$\frac{\partial u}{\partial t} - D \frac{\partial^2 u}{\partial x^2} = f(x, t), \ 0 < x < \ell, \ t > t_0,$$

$$u(x, t_0) = \psi(x), \ 0 < x < \ell,$$

$$\frac{\partial u}{\partial x}(0, t) = 0, \ t > t_0,$$

$$\frac{\partial u}{\partial x}(\ell, t) = 0, \ t > t_0.$$
13. (a) The rate is 
\[-\pi r^2 D \left( \frac{u_\ell - u_0}{\ell} \right).\]
(b) The rate of mass transfer varies inversely with \(\ell\) and directly with the square of \(r\).

2.2 The hanging bar

1. The sum of the forces on the cross-section originally at \(x = \ell\) must be zero. As derived in the text, the internal (elastic) force acting on this cross-section is 
\[-Ak(\ell) \frac{du}{dx}(\ell),\]
while the external traction results in a total force of \(pA\) (force per unit area times area). Therefore, we obtain
\[-Ak(\ell) \frac{du}{dx}(\ell) + pA = 0,\]
or
\[k(\ell) \frac{du}{dx}(\ell) = p.\]
The other boundary condition, of course, is \(u(0) = 0\).

3. (a) The stiffness is the ratio of the internal restoring force to the relative change in length. Therefore, if a bar of length \(\ell\) and cross-sectional area \(A\) is stretched to a length of \(1 + p\) \((0 < p < 1)\) times \(\ell\), then the bar will pull back with a force of \(195pA\) gigaNewtons. Equivalently, this is the force that must be applied to the end of the bar to stretch the bar to a length of \((1 + p)\ell\).

(b) 196/195 m, or approximately 1.005 m.

(c) The BVP is
\[-195 \cdot 10^9 \frac{d^2u}{dx^2} = 0, \quad 0 < x < 1,\]
\[u(0) = 0,\]
\[195 \cdot 10^9 \frac{du}{dx}(1) = 10^9.\]
It is easy to show by direct integration that the solution is \(u(x) = x/195\), and therefore \(u(1) = 1/195\). Since \(u\) is the displacement (in meters), the final position of the end of the bar is \(1 + 1/195\) m, that is, the stretched bar is 196/195 m.

5. The wave equation is
\[(7.9 \cdot 10^3) \frac{\partial^2 u}{\partial t^2} - (1.95 \cdot 10^{11}) \frac{\partial^2 u}{\partial x^2} = f(x,t), \quad 0 < x < 1.\]
The units of \(f\) are \(N/m^3\) (force per unit volume). The units of the first term on the left are
\[\frac{kg \, m}{m^3 \, s^2} = \frac{kgm/s^2}{m^3} = \frac{N}{m^3},\]
and the units of the second term on the left are
\[\frac{N \, 1}{m^3 \, m} = \frac{N}{m^3}.
7. (a) We have 
\[\frac{\partial^2 u}{\partial t^2}(x,t) = -c^2 \theta^2 u(x,t),\]
while 
\[\frac{\partial^2 u}{\partial x^2}(x,t) = -\theta^2 u(x,t).\]
Therefore,
\[\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = -c^2 \theta^2 u + c^2 \theta^2 u = 0.\]
(b) Regardless of the value of \(\theta\), \(u(0,t) = 0\) holds for all \(t\). The only way that \(u(\ell,t) = 0\) can hold for all \(t\) is if \(\sin (\theta \ell) = 0\), so \(\theta\) must be one of the values
\[\theta = \frac{n\pi}{\ell}, \quad n = 0, \pm 1, \pm 2, \ldots.\]
2.3 The wave equation for a vibrating string

1. Units of acceleration (length per time squared).
2. Units of velocity (length per time).
3. The internal force acting on the end of the string at, say, \( x = \ell \), is

\[
T \frac{\partial u}{\partial x}(\ell, t).
\]

(2.1)

If this end can move freely in the vertical direction, force balance implies that (2.1) must be zero.

7. If \( u(x, t) = f(x - ct) \), then

\[
\frac{\partial v}{\partial x}(x, t) = -f'(x - ct), \quad \frac{\partial^2 v}{\partial x^2}(x, t) = -c^2 f''(x - ct),
\]

and hence

\[
\frac{\partial^2 u}{\partial t^2}(x, t) - c^2 \frac{\partial^2 u}{\partial x^2}(x, t) = c^2 f''(x - ct) - c^2 f''(x - ct) = 0,
\]

as desired. Similarly, if \( v(x, t) = f(x + ct) \), then

\[
\frac{\partial v}{\partial x}(x, t) = -f'(x + ct), \quad \frac{\partial^2 v}{\partial x^2}(x, t) = -c^2 f''(x + ct),
\]

and hence

\[
\frac{\partial^2 v}{\partial t^2}(x, t) - c^2 \frac{\partial^2 v}{\partial x^2}(x, t) = c^2 f''(x + ct) - c^2 f''(x + ct) = 0.
\]

2.4 Advection; kinematic waves

1. The solution \( u \) of

\[
\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0, \quad 0 < x < \ell, \quad t > 0,
\]

\[
u(x, 0) = u_0(x), \quad 0 < x < \ell,
\]

\[
u(0, t) = \phi(t), \quad t > 0.
\]

can be found by reasoning similar to that used to solve the pure IVP (see (2.23) and following in the text). The cross-section at point \( x \) at time \( t \) was \( ct \) units to the left at time \( t = 0 \). If \( x - ct > 0 \), then the value of \( u(x, t) \) is given by the initial condition: \( u(x, t) = u_0(x - ct) \). If \( x - ct < 0 \), then \( u(x, t) \) is determined by the boundary condition \( u(0, t) = \phi(t) \), and we must ask: At what time was the cross-section, now at point \( x \) at time \( t \), found at \( x = 0 \)? The answer is \( t_0 \), where \( c(t - t_0) = x \), that is, \( t_0 = t - (1/c)x \). Thus \( u(x, t) = \phi(t - (1/c)x) \) if \( x - ct < 0 \). It is easy to verify by a direct calculation that

\[
u(x, t) = \begin{cases} 
  u_0(x - ct), & x - ct > 0, \\
  \phi \left(t - \frac{1}{c}x\right), & x - ct < 0
\end{cases}
\]

solves the given IBVP.

3. Suppose \( u \) solves

\[
\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty, \quad t > 0,
\]

\[
u(x, 0) = \phi(x), \quad -\infty < x < \infty,
\]

where \( c > 0 \), and suppose that \( \phi(x) = 0 \) for all \( x \leq a \). The solution is \( u(x, t) = \phi(x - ct) \), so \( u(x, t) = 0 \) if \( x - ct \leq a \), that is, if \( t > (x - a)/c \).
Chapter 3

Essential Linear Algebra

3.1 Linear systems as linear operator equations

1. A function of the form \( f(x) = ax + b \) is linear if and only if \( b = 0 \). Indeed, if \( f : \mathbb{R} \to \mathbb{R} \) is linear, then \( f(x) = ax \), where \( a = f(1) \).

3. Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by

\[
 f(x) = \begin{bmatrix}
 x_1^2 + x_2^2 \\
 x_2 - x_1^2 
\end{bmatrix}
\]

If we take \( x = (1,1) \), then \( f(x) = (2,0) \), while 2\( x = (2,2) \) and hence \( f(2x) = (8,-2) \neq 2f(x) \). Thus \( f \) is not linear.

5. (a) Vector space (subspace of \( C[0,1] \)).
(b) Not a vector space; does not contain the zero function. (Subset of \( C[0,1] \), but not a subspace.)
(c) Vector space (subspace of \( C[0,1] \)).
(d) Vector space.
(e) Not a vector space; does not contain the zero polynomial. (Subset of \( P_n \), but not a subspace.)

7. Suppose \( u \in C^1[a,b] \) is any nonzero function. Then

\[
 L(2u) = 2 \frac{d^2u}{dx^2} + (2u)^3 = 2 \frac{du}{dx} + 8u^3,
\]

while

\[
 2Lu = 2 \frac{du}{dx} + 2u^3.
\]

Since \( L(2u) \neq 2Lu \), \( L \) is not linear.

9. Define \( K : C^2[a,b] \to C[a,b] \) by

\[
 K u = \frac{x^2}{d^2u}{dx^2} - 2x \frac{du}{dx} + 3u.
\]

Then \( K \) is linear:

\[
 K(\alpha u) = x^2 \frac{d^2}{dx^2} (\alpha u) - 2x \frac{d}{dx} (\alpha u) + 3(\alpha u)
 = \alpha x^2 \frac{d^2u}{dx^2} - 2 \alpha x \frac{du}{dx} + 3 \alpha u
 = \alpha \left( x^2 \frac{d^2u}{dx^2} - 2x \frac{du}{dx} + 3u \right) = \alpha K u,
\]

\[
 K(u + v) = x^2 \frac{d^2}{dx^2} (u + v) - 2x \frac{d}{dx} (u + v) + 3(u + v)
 = x^2 \left( \frac{d^2u}{dx^2} + \frac{dv}{dx} \right) - 2x \left( \frac{du}{dx} + \frac{dv}{dx} \right) + 3(u + v)
 = \left( x^2 \frac{d^2u}{dx^2} - 2x \frac{du}{dx} + 3u \right) + \left( x^2 \frac{d^2v}{dx^2} - 2x \frac{dv}{dx} + 3v \right) = Ku + Kv.
\]
11. Let \( \rho, c, \) and \( \kappa \) be constants, and define \( L : C^2(\mathbb{R}^2) \to C(\mathbb{R}^2) \) by
\[
Lu = \rho \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2}.
\]
We prove that \( L \) is linear as follows:
\[
L(\alpha u) = \rho \frac{\partial}{\partial t}(\alpha u) - \kappa \frac{\partial^2}{\partial x^2}(\alpha u) = \alpha \rho \frac{\partial u}{\partial t} - \alpha \kappa \frac{\partial^2 u}{\partial x^2} = \alpha \left( \rho \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} \right) = \alpha Lu,
\]
\[
L(u + v) = \rho \frac{\partial}{\partial t}(u + v) - \kappa \frac{\partial^2}{\partial x^2}(u + v) = \rho \left( \frac{\partial u}{\partial t} + \frac{\partial v}{\partial t} \right) - \kappa \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} \right)
= \left( \rho \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} \right) + \left( \rho \frac{\partial v}{\partial t} - \kappa \frac{\partial^2 v}{\partial x^2} \right) = Lu +Lv.
\]
13. (a) The proof is a direct calculation:
\[
A(\alpha x + \beta z) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \alpha \begin{bmatrix} x_1 & x_2 \end{bmatrix} + \beta \begin{bmatrix} z_1 & z_2 \end{bmatrix} \end{bmatrix}
= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \alpha x_1 + \beta z_1 & \alpha x_2 + \beta z_2 \end{bmatrix}
= \begin{bmatrix} a_{11}(\alpha x_1 + \beta z_1) + a_{12}(\alpha x_2 + \beta z_2) \\ a_{21}(\alpha x_1 + \beta z_1) + a_{22}(\alpha x_2 + \beta z_2) \end{bmatrix}
= \alpha \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} + \beta \begin{bmatrix} a_{11}z_1 + a_{12}z_2 \\ a_{21}z_1 + a_{22}z_2 \end{bmatrix}
= \alpha A x + \beta A z.
\]
This shows that the operator defined by \( A \) is linear.

(b) We have
\[
(A(\alpha x + \beta y))_i = \sum_{j=1}^{n} a_{ij}(\alpha x_j + \beta y_j)
\]
and
\[
(\alpha A x + \beta A y)_i = \alpha \sum_{j=1}^{n} a_{ij}x_j + \beta \sum_{j=1}^{n} a_{ij}y_j.
\]
Now,
\[
(A(\alpha x + \beta y))_i = \sum_{j=1}^{n} a_{ij}(\alpha x_j + \beta y_j)
= \sum_{j=1}^{n} (\alpha a_{ij}x_j + \beta a_{ij}y_j)
= \sum_{j=1}^{n} a_{ij}x_j + \beta \sum_{j=1}^{n} a_{ij}y_j
= \alpha \sum_{j=1}^{n} a_{ij}x_j + \beta \sum_{j=1}^{n} a_{ij}y_j
= (\alpha A x + \beta A y)_i,
\]
as desired. The third equality follows from the commutative and associative properties of addition, while the fourth equality follows from the distributive property of multiplication over addition.
3.2 Existence and uniqueness of solutions to $Ax = b$

1. The range of $A$ is
   \[ \{ \alpha(1, -1) : \alpha \in \mathbb{R} \}, \]
   which is a line in the plane. See Figure 3.1.

3. (a) $A$ is nonsingular.
   (b) $A$ is nonsingular.
   (c) $A$ is singular. Every vector in the range is of the form
   \[
   Ax = \begin{bmatrix}
   1 & 1 \\
   1 & 1
   \end{bmatrix}
   \begin{bmatrix}
   x_1 \\
   x_2
   \end{bmatrix}
   = \begin{bmatrix}
   x_1 + x_2 \\
   x_1 + x_2
   \end{bmatrix}.
   \]
   That is, every $y \in \mathbb{R}^2$ whose first and second components are equal lies in the range of $A$. Thus, for example, $Ax = b$ is solvable for
   \[ b = \begin{bmatrix}
   1 \\
   1
   \end{bmatrix}, \]
   but not for
   \[ b = \begin{bmatrix}
   1 \\
   2
   \end{bmatrix}. \]

5. The solution set is not a subspace, since it cannot contain the zero vector ($A0 = 0 \neq b$).

7. The null space of $L$ is the set of all first-degree polynomials:
   \[ \mathcal{N}(L) = \{ u : [a,b] \to \mathbb{R} : u(x) = mx + c \text{ for some } m,c \in \mathbb{R} \}. \]

9. (a) The only functions in $C^2[a,b]$ that satisfy
   \[ -\frac{d^2u}{dx^2} = 0 \]
   are functions of the form $u(x) = C_1x + C_2$. The Neumann boundary condition at the left endpoint implies that $C_1 = 0$, and the Dirichlet boundary condition at the right endpoint implies that $C_2 = 0$. Therefore, only the zero function is a solution to $L_{\bar{m}}u = 0$, and so the null space of $L_{\bar{m}}$ is trivial.
   (b) Suppose $f \in C^1[a,b]$ is given and $u \in C^2_{\bar{m}}[a,b]$ satisfies
   \[ -\frac{d^2u}{dx^2}(x) = f(x), \quad a < x < b. \]
   Integrating once yields, by the fundamental theorem of calculus,
   \[
   -\int_a^x \frac{d^2u}{dx^2}(s) \, ds = \int_a^x f(s) \, ds
   \]
   \[ \Rightarrow -\frac{du}{dx}(x) + \frac{du}{dx}(a) = \int_a^x f(s) \, ds \]
   \[ \Rightarrow \frac{du}{dx}(x) = -\int_a^x f(s) \, ds. \]
The last step follows from the Neumann condition at \( x = a \). We now integrate again:

\[
\int_a^x \frac{du}{dx}(z) \, dz = -\int_a^x f(s) \, ds \, dz
\]

\[
\Rightarrow \quad u(b) - u(x) = -\int_a^x f(s) \, ds \, dz
\]

\[
\Rightarrow \quad u(x) = \int_a^b f(s) \, ds \, dz.
\]

This shows that \( L_{\partial u} = f \) has a unique solution for each \( f \in C[a,b] \).

11. By the fundamental theorem of calculus,

\[
u(x) = \int_a^x f(s) \, ds
\]

satisfies \( Du = f \). However, this solution is not unique; for any constant \( C \),

\[
u(x) = \int_a^x f(s) \, ds + C
\]

is another solution.

### 3.3 Basis and dimension

1. (a) Both equal \((14, 4, -5)\).

(b) Both equal \(\sum_{j=1}^n (v_j)_i x_j\).

3. The given set is not a basis. If \( A \) is the matrix whose columns are the given vectors, then \( N(A) \) is not trivial.

5. There is a typo in the problem statement; the given polynomials are linearly independent:

\[
\begin{align*}
c_1(1 - x + 2x^2) + c_2(1 - 2x^2) + c_3(1 - 3x + 7x^2) &= 0 \\
\Leftrightarrow (c_1 + c_2 + c_3) + (-c_1 - 3c_3)x + (2c_3 - 2c_2 + 7c_3)x^2 &= 0 \\
c_1 + c_2 + c_3 &= 0, \\
-c_1 - 3c_3 &= 0, \\
2c_1 - 2c_2 + 7c_3 &= 0.
\end{align*}
\]

A direct calculation (using Gaussian elimination) shows that this last system of equations has only the trivial solution, and hence the three polynomials form a linearly independent set.

7. We know that \( P_2 \) has dimension 3, and therefore it suffices to show either that the given set of three vectors spans \( P_2 \) or that it is linearly independent. If \( p \in P_2 \), then we write

\[
c_1 = p(x_1), \ c_3 = p(x_3), \ c_2 = p(x_3)
\]

and define the polynomial

\[
q(x) = c_1 L_1(x) + c_2 L_2(x) + c_3 L_3(x).
\]

Then

\[
q(x_1) = c_1 L_1(x_1) + c_2 L_2(x_1) + c_3 L_3(x_1) = c_1 \cdot 1 + c_2 \cdot 0 + c_3 \cdot 0 = c_1 = p(x_1),
\]

and, similarly, \( q(x_2) = p(x_2), \ q(x_3) = p(x_3) \). But then \( p \) and \( q \) are second-degree polynomials that agree at three distinct points, and three points determine a quadratic (just as two points determine a line). Therefore,

\[
p(x) = c_1 L_1(x) + c_2 L_2(x) + c_3 L_3(x),
\]

and we have shown that every \( p \in P_2 \) can be written as a linear combination of \( L_1, L_2, L_3 \). This completes the proof.
9. Let \( L : C^2_N[a, b] \to C[a, b] \) be the second derivative operator. We wish to find a basis for the null space of \( L \). A function \( u \in C^2_N[a, b] \) belongs to the null space of \( L \) if and only if it satisfies
\[
\frac{d^2u}{dx^2}(x) = 0 \quad \text{for all } x \in [a, b], \quad \frac{du}{dx}(0) = \frac{du}{dx}(\ell) = 0.
\]
The only functions satisfying the differential equation are first-degree polynomials, \( u(x) = mx + c \). Moreover, the boundary conditions are satisfied if and only if the slope \( m \) is zero. Thus \( u \) belongs to the null space of \( L \) if and only if \( u(x) = c \) for some \( c \in \mathbb{R} \), that is, if and only if \( u \) is a constant function. A basis is therefore the set \( \{ u_1 \} \), where \( u_1(x) = 1 \) for all \( x \in [a, b] \).

3.4 Orthogonal bases and projections

1. (a) The verification is a direct calculation of \( v_i \cdot v_j \) for the 6 combinations of \( i, j \). For example,
\[
v_1 \cdot v_1 = \frac{1}{\sqrt{3}} \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} \frac{1}{\sqrt{3}} = 1.
\]
\[
v_1 \cdot v_2 = \frac{1}{\sqrt{3}} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} 0 + \frac{1}{\sqrt{3}} \left( -\frac{1}{\sqrt{2}} \right) = 0.
\]
and so forth.

(b)
\[
x = (v_1 \cdot x)v_1 + (v_2 \cdot x)v_2 + (v_3 \cdot x)v_3
= \frac{4}{\sqrt{3}} v_1 + 0v_2 + \left( -\frac{2}{\sqrt{6}} \right) v_3.
\]

3. We have
\[
\|x + y\|^2 = (x + y, x + y)
= (x, x + y) + (y, x + y)
= (x, x) + (x, y) + (y, x) + (y, y)
= (x, x) + 2(x, y) + (y, y)
= \|x\|^2 + 2(x, y) + \|y\|^2.
\]
Therefore,
\[
\|x + y\|^2 = \|x\|^2 + \|y\|^2
\]
if and only if \( (x, y) = 0 \).

5. First of all, if
\[
(y, z) = 0 \quad \text{for all } z \in W,
\]
then since \( w_i \in W \) for each \( i \), we have, in particular,
\[
(y, w_i) = 0, \quad i = 1, 2, \ldots, n.
\]
Suppose, on the other hand, that
\[
(y, w_i) = 0, \quad i = 1, 2, \ldots, n.
\]
If \( z \) is any vector in \( W \), then, since \( \{w_1, w_2, \ldots, w_n\} \) is a basis for \( W \), there exist scalars \( \alpha_1, \alpha_2, \ldots, \alpha_n \) such that
\[
z = \sum_{i=1}^{n} \alpha_i w_i.
\]
We then have
\[
(y, z) = (y, \sum_{i=1}^{n} \alpha_i w_i) \\
= \sum_{i=1}^{n} \alpha_i (y, w_i) \\
= \sum_{i=1}^{n} \alpha_i \cdot 0 \\
= 0.
\]
This completes the proof.

7. The best approximation to the given data is \( y = 2.0109x - 0.0015151 \). See Figure 3.2.

![Figure 3.2: The data from Exercise 3.4.7 and the best linear approximation.](image)

9. The projection of \( g \) onto \( P_2 \) is
\[
\frac{2}{\pi} q_1(x) + 0q_2(x) + \frac{2\sqrt{5}(\pi^2 - 12)}{\pi^3} q_3(x)
\]
or
\[
\frac{2}{\pi} + \frac{10(\pi^2 - 12)}{\pi^3} - 60\frac{\pi^2 - 12}{\pi^3} x + 60\frac{\pi^2 - 12}{\pi^3} x^2.
\]
See Figure 3.3.

![Figure 3.3: The function \( g(x) = \sin(\pi x) \) and its best quadratic approximation over the interval \([0, 1]\) (see Exercise 3.4.9).](image)

### 3.5 Eigenvalues and eigenvectors of a symmetric matrix

1. The eigenvalues of \( A \) are \( \lambda_1 = 200 \) and \( \lambda_2 = 100 \), and the corresponding (normalized) eigenvectors are
\[
\mathbf{u}_1 = \begin{bmatrix} -0.8 \\ 0.6 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}.
\]
3.5. EIGENVALUES AND EIGENVECTORS OF A SYMMETRIC MATRIX

respectively. The solution is

$$x = \frac{b \cdot u_1}{\lambda_1} u_1 + \frac{b \cdot u_2}{\lambda_2} u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$ 

3. The eigenvalues and eigenvectors of $A$ are $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = -1$ and

$$u_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \quad u_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad u_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}.$$ 

The solution of $Ax = b$ is

$$x = \frac{b \cdot u_1}{\lambda_1} u_1 + \frac{b \cdot u_2}{\lambda_2} u_2 + \frac{b \cdot u_3}{\lambda_3} u_3 = \begin{bmatrix} -3/2 \\ 1/2 \\ 5/2 \end{bmatrix}.$$ 

5. The computation of $u_i \cdot b/\lambda_i$ costs $2n$ operations, so computing all $n$ of these ratios costs $2n^2$ operations. Computing the linear combination is then equivalent to another $n$ dot products, so the total cost is

$$2n^2 + n \times (2n - 1) = 4n^2 - n = O(4n^2).$$ 

7. (a) For $k = 2, 3, \ldots, n - 1$, we have

$$\left( Ls^{(j)} \right)_k = \frac{1}{h^2} \left( -\sin((k-1)j\pi h) + 2\sin(kj\pi h) - \sin((k+1)j\pi h) \right)$$

$$= \frac{1}{h^2} \left( -\sin(kj\pi h)\cos(j\pi h) + \cos(kj\pi h)\sin(j\pi h) + 2\sin(kj\pi h) \right.$$ 

$$- \sin(kj\pi h)\cos(j\pi h) - \cos(kj\pi h)\sin(j\pi h) \right)$$

$$= \frac{1}{h^2} \left( 2 - 2\cos(j\pi h) \right) \sin(kj\pi h)$$

$$= \frac{1}{h^2} \left( 2 - 2\cos(j\pi h) \right) s_k^{(j)}.$$ 

Thus

$$\left( Ls^{(j)} \right)_k = \frac{1}{h^2} (2 - 2\cos(j\pi h)) s_k^{(j)}, \quad k = 2, 3, \ldots, n - 1.$$ 

Similar calculations show that this formula holds for $k = 1$ and $k = n$ also. Therefore, $s^{(j)}$ is an eigenvector of $L$ with eigenvalue

$$\lambda_j = \frac{2 - 2\cos(j\pi h)}{h^2}.$$ 

(b) The eigenvalues $\lambda_j$ are all positive and are increasing with the frequency $j$.

(c) The right-hand side $b$ can be expressed as

$$b = \left( s^{(1)} \cdot b \right) s^{(1)} + \left( s^{(2)} \cdot b \right) s^{(2)} + \cdots + \left( s^{(n)} \cdot b \right) s^{(n)},$$

while the solution $x$ of $Lx = b$ is

$$x = \frac{s^{(1)} \cdot x}{\lambda_1} s^{(1)} + \frac{s^{(2)} \cdot x}{\lambda_2} s^{(2)} + \cdots + \frac{s^{(n)} \cdot x}{\lambda_n} s^{(n)}.$$ 

Since $\lambda_j$ increases with $j$, this shows that, in producing $x$, the higher frequency components of $b$ are dampened more than are the lower frequency components of $b$. Thus $x$ is smoother than $b$. 

Chapter 4

Essential Ordinary Differential Equations

4.1 Background

1. Define

\[ x_1 = u, \quad x_2 = \frac{du}{dt}. \]

Then

\[ \frac{dx_1}{dt} = x_2, \]
\[ \frac{dx_2}{dt} = -\frac{1}{2} x_1. \]

3. Define

\[ x_1 = u, \quad x_2 = \frac{du}{dt}, \quad x_3 = \frac{d^2 u}{dt^2}, \quad x_4 = \frac{d^3 u}{dt^3}. \]

Then

\[ \frac{dx_1}{dt} = x_2, \]
\[ \frac{dx_2}{dt} = x_3, \]
\[ \frac{dx_3}{dt} = x_4, \]
\[ \frac{dx_4}{dt} = 2x_3 - x_1 + \sin t. \]

5. Define

\[ x_1 = p, \quad x_2 = \frac{dp}{dt}, \quad x_3 = q, \quad x_4 = \frac{dq}{dt}. \]

Then the first-order system is

\[ \frac{dx_1}{dt} = x_2, \]
\[ m_1 \frac{dx_2}{dt} = f_1(x_1, x_3, x_2, x_4), \]
\[ \frac{dx_3}{dt} = x_4, \]
\[ m_2 \frac{dx_4}{dt} = f_2(x_1, x_3, x_2, x_4). \]

7. Let \( u_1(t) = e^{-t}, \) \( u_2(t) = e^{-2t}. \)
(a) We have
\[
\frac{du_1}{dt}(t) = -e^{-t}, \quad \frac{d^2u_1}{dt^2}(t) = e^{-t},
\]
and therefore
\[
\frac{d^2u_1}{dt^2}(t) + 3\frac{du_1}{dt}(t) + 2u(t) = e^{-t} - 3e^{-t} + 2e^{-t} = 0.
\]
Similarly,
\[
\frac{du_2}{dt}(t) = -2e^{-2t}, \quad \frac{d^2u_2}{dt^2}(t) = 4e^{-2t},
\]
and thus
\[
\frac{d^2u_2}{dt^2}(t) + 3\frac{du_2}{dt}(t) + 2u(t) = 4e^{-2t} - 6e^{-2t} + 2e^{-2t} = 0.
\]
(b) The Wronskian of \(u_1, u_2\) is
\[
W(t) = \begin{vmatrix} u_1(t) & u_2(t) \\ \frac{du_1}{dt}(t) & \frac{du_2}{dt}(t) \end{vmatrix} = \begin{vmatrix} e^{-t} & 2e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{vmatrix} = -e^{-3t}.
\]
Since \(W(t) \neq 0\) for all \(t\), we see that \(\{u_1, u_2\}\) is linearly independent.

9. (a) Let \(u_1, u_2\) be two functions defined on an interval, let \(t_0\) be a point in that interval, and suppose
\[
\begin{vmatrix} u_1(t_0) & u_2(t_0) \\ \frac{du_1}{dt}(t_0) & \frac{du_2}{dt}(t_0) \end{vmatrix} \neq 0.
\]
We wish to prove that \(\{u_1, u_2\}\) is linearly independent. We argue by contradiction, and suppose \(\{u_1, u_2\}\) is linearly dependent. Then there exist \(c_1, c_2 \in \mathbb{R}\) such that \(c_1u_1(t) + c_2u_2(t) = 0\) for all \(t\) in the given interval, which in turn implies that \(c_1\frac{du_1}{dt}(t) + c_2\frac{du_2}{dt}(t) = 0\) for all \(t\). But then, taking \(t = t_0\), we obtain
\[
c_1u_1(t_0) + c_2u_2(t_0) = 0,
\]
\[
c_1\frac{du_1}{dt}(t_0) + c_2\frac{du_2}{dt}(t_0) = 0.
\]
In matrix-vector form, this system is
\[
\begin{bmatrix} u_1(t_0) & u_2(t_0) \\ \frac{du_1}{dt}(t_0) & \frac{du_2}{dt}(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]
Since \((c_1, c_2) \neq (0, 0)\), this implies that the matrix
\[
\begin{bmatrix} u_1(t_0) & u_2(t_0) \\ \frac{du_1}{dt}(t_0) & \frac{du_2}{dt}(t_0) \end{bmatrix}
\]
is singular, contradicting the assumption that its determinant is nonzero. This contradiction completes the proof.

(b) Let \(u_1(t) = \cos(t)\), \(u_2(t) = \cos(2t)\). Then
\[
\begin{vmatrix} u_1(t) & u_2(t) \\ \frac{du_1}{dt}(t) & \frac{du_2}{dt}(t) \end{vmatrix} = \begin{vmatrix} \cos(t) & \cos(2t) \\ -\sin(t) & -2\sin(2t) \end{vmatrix} = \cos(2t)\sin(t) - 2\cos(t)\sin(2t).
\]
We see that \(W(\pi/2) = -1 \neq 0\), and hence, by part (a), \(\{u_1, u_2\}\) is linearly independent. Nevertheless, \(W(0) = 0\), which shows that the fact that \(\{u_1, u_2\}\) is linearly independent does not imply that \(W(t) \neq 0\) for all \(t\), unless \(u_1\) and \(u_2\) are both solutions to the same second-order linear differential equation.
4.2 Solutions to some simple ODEs

1. (a) We first note that the zero function is a solution of (4.9), so $S$ is nonempty. If $u, v \in S$ and $\alpha, \beta \in \mathbb{R}$, then

$$w = \alpha u + \beta v \text{ satisfies } a \frac{d^2 w}{dt^2} + b \frac{dw}{dt} + cw = a \frac{d^2}{dt^2} [\alpha u + \beta v] + b \frac{d}{dt} [\alpha u + \beta v] + c(\alpha u + \beta v)$$

$$= a \left( \frac{d^2 u}{dt^2} + b \frac{du}{dt} + \frac{d^2 v}{dt^2} + b \frac{dv}{dt} \right) + c(\alpha u + \beta v)$$

$$= \alpha \left( \frac{d^2 u}{dt^2} + b \frac{du}{dt} + cu \right) + \beta \left( \frac{d^2 v}{dt^2} + b \frac{dv}{dt} + cv \right)$$

$$= \alpha \cdot 0 + \beta \cdot 0 = 0.$$ 

Thus $w$ is also a solution of (4.9), which shows that $S$ is a subspace.

(b) As explained in the text, no matter what the values of $a, b, c$ ($a \neq 0$), the solution space is spanned by two functions. Thus $S$ is finite-dimensional and it has dimension at most 2. Moreover, it is easy to see that each of the sets $\{e^{r_1 t}, e^{r_2 t}\}$ ($r_1 \neq r_2$), $\{e^{\alpha t} \cos (\lambda t), e^{\alpha t} \sin (\lambda t)\}$, and $\{e^{rt}, te^{rt}\}$ is linearly independent, since a set of two functions is linearly dependent if and only if one of the functions is a multiple of the other. Thus, in every case, $S$ has a basis with two functions, and so $S$ is two-dimensional.

3. Suppose that the characteristic polynomial of (4.9) has a single, repeated root $r = -b/(2a)$ (so $b^2 - 4ac = 0$).

Then, as shown in the text,

$$u(t) = c_1 e^{rt} + c_2 t e^{rt}$$

is a solution of (4.9) for each choice of $c_1, c_2$. We wish to show that, given $k_1, k_2 \in \mathbb{R}$, there is a unique choice of $c_1, c_2$ such that $u(0) = k_1, du/dt(0) = k_2$. We have

$$\frac{du}{dt}(t) = r c_1 e^{rt} + c_2 e^{rt} + r c_2 e^{rt}.$$ 

Therefore, the equations $u(0) = k_1, du/dt(0) = k_2$ simplify to

$$\begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix} c = k.$$ 

Since the coefficient matrix is obviously nonsingular (regardless of the value of $r$), there is a unique solution $c_1, c_2$ for each $k_1, k_2$.

5. (a) The only solution is $u(x) = 0$.

(b) The only solution is $u(x) = 0$.

(c) The function $u(x) = \sin (\pi x)$ is a nonzero solution. It is not unique, since any multiple of it is another solution.

7. By the product rule and the fundamental theorem of calculus,

$$\frac{d}{dt} \left[ \int_{t_0}^{t} e^{a(t-s)} f(s) \, ds \right] = \frac{d}{dt} \left[ e^{at} \int_{t_0}^{t} e^{-as} f(s) \, ds \right] = a e^{at} \int_{t_0}^{t} e^{-as} f(s) \, ds + e^{at} e^{-at} f(t)$$

$$= a \int_{t_0}^{t} e^{a(t-s)} f(s) \, ds + f(t).$$

Therefore, with

$$u(t) = u_0 e^{a(t-t_0)} + \int_{t_0}^{t} e^{a(t-s)} f(s) \, ds,$$

we have

$$\frac{du}{dt}(t) = a u_0 e^{a(t-t_0)} + \int_{t_0}^{t} e^{a(t-s)} f(s) \, ds + f(t)$$

$$= a u(t) + f(t).$$

Thus $u$ satisfies the differential equation. We also have

$$u(t_0) = u_0 e^{a(t_0-t_0)} + \int_{t_0}^{t_0} e^{a(t-s)} f(s) \, ds = u_0 + 0 = u_0,$$

so the initial condition holds as well.
9. \( u(t) = \frac{1}{2} \sin^2(t) \)

11. \( u(t) = \frac{1}{2} (e^{-t} + \cos(t) + \sin(t)) \)

13. (a) The solution is \( x(t) = e^{t^2/2} \).

(b) The solution is \( x(t) = (1 + \cos(1) - \cos(t))/t^3 \).

(c) The solution is \( y(t) = 1/2 + (1/2)e^{-t^2} \).

15. Suppose we substitute \( u(t) = c_1(t)u_1(t) + c_2(t)u_2(t) \) into

\[
\frac{d^2u}{dt^2} + b\frac{du}{dt} + cu = f(t),
\]

where \( u_1, u_2 \) are solutions of the homogeneous version of this ODE, and assume that

\[
\frac{dc_1}{dt}u_1 + \frac{dc_2}{dt}u_2 = 0.
\]

We then have

\[
\frac{du}{dt} = c_1\frac{du_1}{dt} + c_2\frac{du_2}{dt},
\]

\[
\frac{d^2u}{dt^2} = c_1\frac{d^2u_1}{dt^2} + c_2\frac{d^2u_2}{dt^2} + \frac{dc_1}{dt}\frac{du_1}{dt} + \frac{dc_2}{dt}\frac{du_2}{dt}.
\]

Substituting into the ODE, we obtain

\[
a\left(c_1\frac{d^2u_1}{dt^2} + c_2\frac{d^2u_2}{dt^2} + \frac{dc_1}{dt}\frac{du_1}{dt} + \frac{dc_2}{dt}\frac{du_2}{dt}\right) + b\left(c_1\frac{du_1}{dt} + c_2\frac{du_2}{dt}\right) + c(c_1u_1 + c_2u_2) = f(t)
\]

\[
\Rightarrow c_1\left(\frac{d^2u_1}{dt^2} + b\frac{du_1}{dt} + cu_1\right) + c_2\left(\frac{d^2u_2}{dt^2} + b\frac{du_2}{dt} + cu_2\right) + a\left(\frac{dc_1}{dt}\frac{du_1}{dt} + \frac{dc_2}{dt}\frac{du_2}{dt}\right) = f(t)
\]

\[
\Rightarrow a\left(\frac{dc_1}{dt}\frac{du_1}{dt} + \frac{dc_2}{dt}\frac{du_2}{dt}\right) = f(t)
\]

\[
\Rightarrow \frac{dc_1}{dt}\frac{du_1}{dt} + \frac{dc_2}{dt}\frac{du_2}{dt} = a^{-1}f(t),
\]

as desired. Notice that we have used the fact that \( u_1, u_2 \) solve the homogenous ODE:

\[
a\frac{d^2u_1}{dt^2} + b\frac{du_1}{dt} + cu_1 = 0, \quad a\frac{d^2u_2}{dt^2} + b\frac{du_2}{dt} + cu_2 = 0.
\]

17. Let us define \( v(s) = u(e^s) \); then \( u(t) = v(\ln(t)) \). It follows that

\[
\frac{du}{dt}(t) = \frac{1}{t} \frac{dv}{ds}(\ln(t)), \quad \frac{d^2u}{dt^2}(t) = \frac{1}{t^2} \frac{d^2v}{ds^2}(\ln(t)) - \frac{1}{t^2} \frac{dv}{ds}(\ln(t)).
\]

It follows that

\[
t^2\frac{d^2u}{dt^2}(t) + at\frac{du}{dt}(t) + bu(t) = 0
\]

\[
\Rightarrow \frac{d^2v}{ds^2}(\ln(t)) - \frac{dv}{ds}(\ln(t)) + a\frac{dv}{ds}(\ln(t)) + bv(\ln(t)) = 0
\]

\[
\Rightarrow \frac{d^2v}{ds^2}(\ln(t)) + (a - 1)\frac{dv}{ds}(\ln(t)) + bv(\ln(t)) = 0
\]

\[
\Rightarrow \frac{d^2v}{ds^2}(s) + (a - 1)\frac{dv}{ds}(s) + bv(s) = 0.
\]

Thus the change of variables \( t = e^s \) transforms the Euler equation into a constant coefficient ODE.
4.3 Linear systems with constant coefficients

1. The solution is

\[ x(t) = \frac{1}{\sqrt{3}} u_1 - \frac{1}{\sqrt{2}} e^{-t} u_2 + \frac{1}{\sqrt{6}} e^{-2t} u_3 \]

\[ = \begin{bmatrix} \frac{1}{3} - \frac{1}{2} e^{-t} + \frac{1}{6} e^{-2t} \\ \frac{1}{3} - \frac{1}{2} e^{-t} + \frac{1}{6} e^{-2t} \\ \frac{1}{3} + \frac{1}{2} e^{-t} + \frac{1}{6} e^{-2t} \end{bmatrix}. \]

3. The general solution is

\[ x(t) = \begin{bmatrix} c_1 e^t + c_2 e^{-1} \\ c_1 e^t - c_2 e^{-1} \end{bmatrix}. \]

5. \( x_0 \) must be a multiple of the vector \((1, -1)\).

7.

\[ x(t) = \frac{1}{60} \begin{bmatrix} 55 - 3e^{-2t} - 45e^{-t} + 20t - 7\cos(t) - 11\sin(t) \\ 10 + 6e^{-2t} + 20t - 16\cos(t) - 8\sin(t) \\ -5 - 3e^{-2t} + 45e^{-1} + 20t - 37\cos(t) + 19\sin(t) \end{bmatrix} \]

9. (a) The solution is

\[ x(t) = \frac{3}{2} e^{-t} + \frac{1}{2} e^{3t}, \quad y(t) = \frac{3}{2} e^{-t} - \frac{1}{2} e^{3t}. \]

The population of the first species \((x(t))\) increases exponentially, while the population of the second species \((y(t))\) goes to zero in finite time \(y(t) = 0\) at \(t = \ln(3)/4\). Thus the second species becomes extinct, while the first species increases without bound.

(b) If the initial populations are \(x(0) = r, \ y(0) = s\), then the solution to the IVP is

\[ x(t) = \frac{1}{2} \left( (r+s)e^{-t} + (r-s)e^{3t} \right), \quad y(t) = \frac{1}{2} \left( (r+s)e^{-t} + (s-r)e^{3t} \right). \]

Therefore, if \(r = s\), both populations decay to zero exponentially; that is, both species die out.

11. Assume \(v(t; s)\) satisfies

\[ \frac{d^2 v}{dt^2} + \theta^2 v = 0, \quad v(0; s) = 0, \quad \frac{dv}{dt}(0; s) = 0 \]

for all \(s\), and define

\[ u(t) = \int_0^t v(t - s; s) \, ds. \]

Then

\[ \frac{du}{dt}(t) = v(0; t) + \int_0^t \frac{dv}{dt}(t - s; s) \, ds = \int_0^t \frac{dv}{dt}(t - s; s) \, ds, \]

\[ \frac{d^2 u}{dt^2}(t) = \frac{dv}{dt}(0; t) + \int_0^t \frac{d^2 v}{dt^2}(t - s; s) \, ds = f(t) + \int_0^t \frac{d^2 v}{dt^2}(t - s; s) \, ds \]

(using the fact that \(v(0; t) = 0\) and \(\frac{dv}{dt}(0; t) = 0\) for all \(t\)). We then have

\[ \frac{d^2 u}{dt^2}(t) + \theta^2 u(t) = f(t) + \int_0^t \frac{d^2 v}{dt^2}(t - s; s) \, ds + \theta^2 \int_0^t v(t - s; s) \, ds \]

\[ = f(t) + \int_0^t \left\{ \frac{d^2 v}{dt^2}(t - s; s) + \theta^2 v(t - s; s) \right\} \, ds \]

\[ = f(t) \]

since

\[ \frac{d^2 v}{dt^2}(t - s; s) + \theta^2 v(t - s; s) = 0 \text{ for all } s. \]

Also,

\[ u(t) = \int_0^t v(t - s; s) \, ds = 0, \quad \frac{du}{dt}(t) = \int_0^t \frac{dv}{dt}(t - s; s) \, ds = 0, \]

and thus \(u\) satisfies both the differential equations and the initial conditions.
13. Suppose that, for all $s$, $v(t; s)$ satisfies the ODE
\[
\frac{d^k u}{dt^k} + a_{k-1}(t)\frac{d^{k-1} u}{dt^{k-1}} + \cdots + a_1(t)\frac{du}{dt} + a_0(t)u = 0
\]
and also the initial conditions
\[
u(s) = 0, \quad \frac{du}{dt}(s) = 0, \quad \ldots, \quad \frac{d^{k-2} u}{dt^{k-2}}(s) = 0, \quad \frac{d^{k-1} u}{dt^{k-1}}(s) = f(s).
\]
Define
\[
u(t) = \int_0^t v(t; s) \, ds.
\]
Then
\[
\frac{du}{dt}(t) = v(t; t) + \int_0^t \frac{dv}{dt}(t; s) \, ds = \int_0^t \frac{dv}{dt}(t; s) \, ds
\]
(since $v(t; t) = 0$ for all $t$). Similarly,
\[
\frac{d^2 u}{dt^2}(t) = \frac{dv}{dt}(t; t) + \int_0^t \frac{d^2 v}{dt^2}(t; s) \, ds = \int_0^t \frac{d^2 v}{dt^2}(t; s) \, ds,
\]
and so on.
\[
\frac{d^{k-1} u}{dt^{k-1}}(t) = \frac{d^{k-2} v}{dt^{k-2}}(t; t) + \int_0^t \frac{d^{k-1} v}{dt^{k-1}}(t; s) \, ds = \int_0^t \frac{d^{k-1} v}{dt^{k-1}}(t; s) \, ds
\]
and finally
\[
\frac{d^k u}{dt^k}(t) = \frac{d^{k-1} v}{dt^{k-1}}(t; t) + \int_0^t \frac{d^k v}{dt^k}(t; s) \, ds = f(t) + \int_0^t \frac{d^k v}{dt^k}(t; s) \, ds
\]
We then see that
\[
\frac{d^k u}{dt^k} + a_{k-1}(t)\frac{d^{k-1} u}{dt^{k-1}} + \cdots + a_1(t)\frac{du}{dt} + a_0(t)u
\]
\[= f(t) + \int_0^t \left\{ \frac{d^k v}{dt^k}(t; s) + a_{k-1}(t)\frac{d^{k-1} v}{dt^{k-1}}(t; s) + \cdots + a_1(t)\frac{dv}{dt}(t; s) + a_0(t)v(t; s) \right\} \, ds
\]
\[= f(t),
\]
by assumption. Also,
\[
\frac{d^j u}{dt^j}(0) = \int_0^0 \frac{d^j v}{dt^j}(t; s) \, ds = 0, \quad j = 0, 1, \ldots, k - 1.
\]
15. The solution to
\[
\frac{d^2 u}{dt^2} + 3\frac{du}{dt} + 2u = 0,
\]
\[u(0) = 0,
\]
\[\frac{du}{dt}(0) = f(s)
\]
is given by $v(t; s) = (e^{-t} - e^{-2t})f(s)$. Therefore, by Duhamel’s principle, the solution of
\[
\frac{d^2 u}{dt^2} + 3\frac{du}{dt} + 2u = f(t),
\]
\[u(0) = 0,
\]
\[\frac{du}{dt}(0) = 0
\]
is
\[
u(t) = \int_0^t v(t - s; s) \, ds = \int_0^t \left( e^{-(t-s)} - e^{-2(t-s)} \right) f(s) \, ds.
\]
4.4 Numerical methods for initial value problems

1. (a) Four steps of Euler’s method yield an estimate of 0.71969.
   (b) Two steps of the improved Euler method yield an estimate of 0.80687.
   (c) One step of the RK4 method yields an estimate of 0.82380.

Euler’s method gives no correct digits, the improved Euler method gives one correct digit, and RK4 gives three correct digits. Each of the methods evaluated $f(t,u)$ four times.

3. (a) Let $u_1 = x$, $u_2 = dx/dt$, $u_3 = y$, $u_4 = dy/dt$. Then the system is

\[
\begin{align*}
\frac{du_1}{dt} &= u_2, \\
\frac{du_2}{dt} &= u_1 + 2u_4 - \frac{\mu_2(u_1 + \mu_1)}{r_1(u_1, u_3)^3} - \frac{\mu_1(u_1 - \mu_2)}{r_2(u_1, u_3)^3}, \\
\frac{du_3}{dt} &= u_4, \\
\frac{du_4}{dt} &= -2u_2 + u_3 - \frac{\mu_2 u_3}{r_1(u_1, u_3)^3} - \frac{\mu_1 u_3}{r_2(u_1, u_3)^3}.
\end{align*}
\]

(b) The routine `ode45` from MATLAB (version 5.3) required 421 steps to produce a graph with the ending point apparently coinciding with the initial value. The graph of $y(t)$ versus $x(t)$ is given in Figure 4.1. The graphs of $x(t)$ and $y(t)$, together with the times steps, are given in Figure 4.2. They show, not surprisingly, that the time step is forced to be small precisely when the coordinates are changing rapidly.

![Figure 4.1: The orbit of the satellite in Exercise 4.4.3.](image)

![Figure 4.2: The coordinates of the satellite in Exercise 4.4.3 (top two graphs) with the step lengths taken (bottom graph).](image)

(c) The minimum step size taken by `ode45` was $3.28 \cdot 10^{-4}$, and using this step length over the entire interval of $[0,T]$ would require almost 19000 steps. This is to be compared to the 421 steps taken by the adaptive algorithm. (Note: The exact results for minimum step size, etc., will differ according to the algorithm and tolerances used.)
5. (a) We first note that \(a + (t_1 - t_0) \leq t \leq b + (t_1 - t_0)\) if and only if \(a \leq t - (t_1 - t_0) \leq b\), so the function \(v\) is well-defined. In fact,

\[
\{v(t) : t \in [a + (t_1 - t_0), b + (t_1 - t_0)]\} = \{u(t) : t \in [a + (t_1 - t_0), b + (t_1 - t_0)]\} = \{u(t) : t \in [a, b]\}
\]

(just replace \(t - (t_1 - t_0)\) by \(t\) in the last step).

We have

\[
\frac{dv}{dt}(t) = \frac{d}{dt}[u(t - (t_1 - t_0))] = \frac{du}{dt}(t - (t_1 - t_0)) = f(u(t - (t_1 - t_0))) = f(v(t)),
\]

so \(v\) satisfies the ODE. Finally,

\[v(t_1) = u(t_1 - (t_1 - t_0)) = u(t_0) = u_0.\]

Therefore \(v\) is a solution to the given IVP.

(b) Consider the IVP

\[
\frac{du}{dt} = t, \quad u(0) = 1
\]

which has solution \(u(t) = t^2/2 + 1\). The IVP

\[
\frac{dv}{dt} = t, \quad v(1) = 1
\]

has solution \(v(t) = t^2/2 + 1/2\), which is not equal to

\[u(t - (1 - 0)) = u(t - 1) = \frac{1}{2}(t - 1)^2 + 1.\]

7. The exact solution is

\[x(t) = \begin{cases} 2 - 2e^{-t}, & 0 < t < \ln 2, \\ \frac{1}{2}e^t, & t > \ln 2. \end{cases}\]

(a) The following errors were obtained at \(t = 0.5\):

<table>
<thead>
<tr>
<th>(\Delta t)</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>2.4332e-05</td>
</tr>
<tr>
<td>1/8</td>
<td>1.3697e-06</td>
</tr>
<tr>
<td>1/16</td>
<td>8.1251e-08</td>
</tr>
<tr>
<td>1/32</td>
<td>4.9475e-09</td>
</tr>
<tr>
<td>1/64</td>
<td>3.0522e-10</td>
</tr>
<tr>
<td>1/128</td>
<td>1.8952e-11</td>
</tr>
</tbody>
</table>

By inspection, we see that as \(\Delta t\) is cut in half, the error is reduced by a factor of approximately 16, as expected for \(O(\Delta t^4)\) convergence.

(b) The following errors were obtained at \(t = 2.0\):
4.5 STIFF SYSTEMS OF ODES

The error definitely does not exhibit \( O(\Delta t^4) \) convergence to zero. The reason is the lack of smoothness of the function \( 1 + |x - 1| \); the rate of convergence given in the text only applies to ODEs defined by smooth functions. When integrating from \( t = 0 \) to \( t = 0.5 \), the nonsmoothness of the right-hand side is not encountered since \( x(t) < 1 \) on this interval. This explains why we observed good convergence in the first part of this problem.

### 4.5 Stiff systems of ODEs

1. (a) The exact solution is
   \[
   x(t) = \frac{1}{2} \left[ e^{-t} + e^t \right].
   \]
   (b) The norm of the error is approximately 0.089103.
   (c) The norm of the error is approximately 0.10658.
2. The largest value is \( \Delta t = 0.04 \).
3. (a) Applying the methods of Section 4.3, we find the solution
   \[
   x(t) = \frac{1}{2} \left[ 3e^{-t} - e^{-100t} \right].
   \]
   (b) By trial and error, we find that \( \Delta t \leq 0.02 \) is necessary for stability. Specifically, integrating from \( t = 0 \) to \( t = 1, n = 49 \) (i.e. \( \Delta t = 1/49 \)) yields
   \[
   \| x_{49} \| > \| x_0 \|,
   \]
   while \( n \geq 50 \) yields
   \[
   \| x_n \| \leq \| x_0 \|.
   \]
   (c) With \( x_{i+1} = x_i + \Delta t A x_i \) and \( y_{1(i)} = u_1 \cdot x_i \), we have
   \[
   u_1 \cdot x_{i+1} = u_1 \cdot x_i + \Delta t u_1 \cdot A x_i,
   \]
   \[
   \Rightarrow y_{1(i+1)} = y_{1(i)} + \Delta t \lambda_1 y_{1(i)}
   \]
   \[
   \Rightarrow y_{2(i+1)} = (1 + \Delta t \lambda_1) y_{2(i)}.
   \]
   A similar calculation shows that
   \[
   y_{2(i+1)} = (1 + \Delta t \lambda_1) y_{2(i)}.
   \]
   For stability, we need
   \[
   |1 + \Delta t \lambda_1| \leq 1,
   \]
   \[
   |1 + \Delta t \lambda_2| \leq 1.
   \]
   Since the eigenvalues of \( A \) are \(-1, -100\), it is not hard to see that the upper bound for \( \Delta t \) is \( \Delta t \leq 0.02 \), just as was determined by experiment.
4. (d) For the backward Euler method, a similar calculation shows that
   \[
   y_{1(i+1)} = (1 - \Delta t \lambda_1)^{-1} y_{1(i)},
   \]
   \[
   y_{2(i+1)} = (1 - \Delta t \lambda_2)^{-1} y_{2(i)}.
   \]
   Stability is guaranteed for any \( \Delta t \), since
   \[
   |1 - \Delta t \lambda_i|^{-1} < 1
   \]
   for every \( \Delta t > 0 \).

<table>
<thead>
<tr>
<th>( \Delta t )</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>5.0774e-03</td>
</tr>
<tr>
<td>1/8</td>
<td>3.2345e-03</td>
</tr>
<tr>
<td>1/16</td>
<td>3.1896e-04</td>
</tr>
<tr>
<td>1/32</td>
<td>1.0063e-04</td>
</tr>
<tr>
<td>1/64</td>
<td>8.9026e-06</td>
</tr>
<tr>
<td>1/128</td>
<td>3.4808e-06</td>
</tr>
</tbody>
</table>
Chapter 5

Boundary Value Problems in Statics

5.1 The analogy between BVPs and linear algebraic systems

1. (a) Since $M_D$ is linear, it suffices to show that the null space of $M_D$ is trivial. If $M_Du = 0$, then $du/dx = 0$, that is, $u$ is a constant function: $u(x) = c$. But then the condition $u(0) = 0$ implies that $c = 0$, and so $u$ is the zero function. Thus $\mathcal{N}(M_D) = \{0\}$.

(b) If $u \in C^1[0, \ell]$ and $M_Du = f$, then we have

$$\frac{du}{dx}(x) = f(x), \quad 0 < x < \ell \quad \text{and} \quad u(0) = 0,$$

which imply that

$$u(x) = \int_0^x f(s) \, ds.$$

But we also must have $u(\ell) = 0$, which implies that

$$\int_0^\ell f(s) \, ds = 0.$$

If $f \in C[0, \ell]$ does not satisfy this condition, then it is impossible for $M_Du = f$ to have a solution.

3. (a) If $v, w \in S$ are both solutions of $Lu = f$, then $Lv = Lw$, or (by linearity) $L(v - w) = 0$. Therefore $v - w \in \mathcal{N}(L)$. But, since $S$ is a subspace, $v - w$ is also in $S$. If the only function in both $\mathcal{N}(L)$ and $S$ is the zero function, then $v - w$ must be the zero function, that is, $v$ and $w$ must be the same function. Therefore, if $\mathcal{N}(L) \cap S = \{0\}$, then $Lu = f$ can have at most one solution for any $f$.

(b) We have already seen that $Lu = f$ has a solution for any $f \in C[0, \ell]$ (see the discussion immediately preceding Example 5.1). We will use the result from the first part of this exercise to show that the solution is unique.

The null space of $L$ is the space of all first degree polynomials:

$$\mathcal{N}(L) = \{u : [0, \ell] \to \mathbb{R} : u(x) = ax + b \text{ for some } a, b \in \mathbb{R}\}.$$

i. Suppose $u \in \mathcal{N}(L) \cap S$. Then $u(x) = ax + b$ for some $a, b \in \mathbb{R}$ and

$$\frac{du}{dx} \left( \frac{\ell}{2} \right) = 0 \Rightarrow a = 0.$$

Then $u(x) = b$, and

$$\int_0^\ell u(x) \, dx = 0 \Rightarrow b\ell = 0 \Rightarrow b = 0.$$

Therefore $u$ is the zero function, and so $\mathcal{N}(L) \cap S = \{0\}$. The uniqueness property then follows from the first part of this exercise.
CHAPTER 5. BOUNDARY VALUE PROBLEMS IN STATICS

ii. Suppose \( u \in \mathcal{N}(L) \cap S \). Then \( u(x) = ax + b \) for some \( a, b \in \mathbb{R} \) and

\[
    u(0) = 0 \Rightarrow b = 0.
\]

Then \( u(x) = ax \), and

\[
    \frac{du}{dx}(\ell) = 0 \Rightarrow a = 0.
\]

Therefore \( u \) is the zero function, and so \( \mathcal{N}(L) \cap S = \{0\} \). The uniqueness property then follows from the first part of this exercise.

5. The condition

\[
    \int_0^\ell \left( \frac{du}{dx}(x) \right)^2 \, dx = 0 \tag{5.1}
\]

implies that \( du/dx \) is zero, and hence that \( u \) is constant. The boundary conditions on \( u \) would then imply that \( u \) is the zero function. But, by assumption, \( u \) is nonzero (we assumed that \( (u, u) = 1 \)). Therefore, (5.1) cannot hold.

7. (a) If \( L_{\hat{\alpha}}u = 0 \), then \( u \) has the form \( u(x) = ax + b \). The first boundary condition, \( du/dx(0) = 0 \), implies that \( a = 0 \), and then the second boundary condition, \( u(\ell) = 0 \), yields \( b = 0 \). Therefore, \( u \) is the zero function and \( \mathcal{N}(L_{\hat{\alpha}}) \) is trivial.

(b) For any \( f \in C[0, \ell] \), the function

\[
    u(x) = \int_x^\ell \int_0^z f(s) \, ds \, dz
\]

belongs to \( C^2_{\hat{\alpha}}[0, \ell] \) and satisfies

\[
    -\frac{d^2u}{dx^2}(x) = f(x), \quad 0 < x < \ell.
\]

This shows that \( L_{\hat{\alpha}}u = f \) has a solution for any \( f \in C[0, \ell] \), and therefore \( \mathcal{R}(L_{\hat{\alpha}}) = C[0, \ell] \).

(c) Suppose \( u, v \in C^2_{\hat{\alpha}}[0, \ell] \). Then

\[
    (L_{\hat{\alpha}}u, v) = -\int_0^\ell \frac{d^2u}{dx^2}(x)v(x) \, dx
\]

\[
    = -\left[ \frac{du}{dx}(x)v(x) \right]_0^\ell + \int_0^\ell \frac{du}{dx}(x)\frac{dv}{dx}(x) \, dx
\]

\[
    = \int_0^\ell \frac{du}{dx}(x)\frac{dv}{dx}(x) \, dx (\text{since } v(\ell) = 0, \ du/dx(0) = 0)
\]

\[
    = \left[ u(x)\frac{dv}{dx}(x) \right]_0^\ell - \int_0^\ell u(x)\frac{d^2v}{dx^2}(x) \, dx
\]

\[
    = -\int_0^\ell u(x)\frac{d^2v}{dx^2}(x) \, dx (\text{since } u(\ell) = 0, \ dv/dx(0) = 0)
\]

\[
    = (u, L_{\hat{\alpha}}v).
\]

Thus \( L_{\hat{\alpha}} \) is symmetric.

(d) Suppose \( \lambda \) is an eigenvalue of \( L_{\hat{\alpha}} \) with corresponding eigenfunction \( u \), and assume that \( u \) has been normalized so that \( (u, u) = 1 \). Then

\[
    \lambda = \lambda(u, u) = (\lambda u, u) = (L_{\hat{\alpha}}u, u)
\]

\[
    = -\int_0^\ell \frac{d^2u}{dx^2}(x)u(x) \, dx
\]

\[
    = -\left[ \frac{du}{dx}(x)u(x) \right]_0^\ell + \int_0^\ell \left( \frac{du}{dx}(x) \right)^2 \, dx
\]

\[
    = \int_0^\ell \left( \frac{du}{dx}(x) \right)^2 \, dx (\text{since } u(\ell) = 0, \ du/dx(0) = 0)
\]

\[
    > 0.
\]

The last step follows because every nonzero function in \( C^2_{\hat{\alpha}}[0, \ell] \) has a nonzero derivative.
9. Define \( u(x) = x(1 - x), \ v(x) = x^2(1 - x) \). Then a direct calculation shows that

\[
(Mu, v) = \frac{7}{30},
\]

but

\[
(u, Mv) = \frac{4}{15}.
\]

11. (a) Suppose \( u, v \in C^2_0[0, \ell] \). Then

\[
\begin{align*}
(L_Ru, v) &= -\kappa \int_0^\ell \frac{d^2 u}{dx^2}(x)v(x) \, dx \\
&= -\left[ \kappa \frac{du}{dx}(x)v(x) \right]_0^\ell + \kappa \int_0^\ell \frac{du}{dx}(x) \frac{dv}{dx}(x) \, dx \\
&= -\left[ \kappa \frac{du}{dx}(x)v(x) \right]_0^\ell + \left[ \kappa u(x) \frac{dv}{dx}(x) \right]_0^\ell - \kappa \int_0^\ell u(x) \frac{d^2 v}{dx^2}(x) \, dx \\
&= -\kappa \frac{du}{dx}(\ell)v(\ell) + \kappa \frac{du}{dx}(0)v(0) + u(\ell)\kappa \frac{dv}{dx}(\ell) - u(0)\kappa \frac{dv}{dx}(0)
\end{align*}
\]

Applying the boundary conditions on \( u, v \), we obtain

\[
\frac{Ma}{\alpha} + \frac{ab}{\alpha \ell} = 0,
\]

so \( (L_Ru, v) = (u, L_Rv) \), as desired.

(b) If \( L_Ru = 0 \), then \( u \) must be a first degree polynomial: \( u(x) = ax + b \). The boundary conditions imply that \( a \) and \( b \) must satisfy the following equations:

\[
\begin{align*}
-ka + ab &= 0, \\
(\alpha \ell + \kappa)a + ab &= 0.
\end{align*}
\]

The determinant is computed as follows:

\[
\begin{vmatrix}
-\kappa & \alpha \\
\alpha \ell + \kappa & \alpha
\end{vmatrix} = -\alpha(2\kappa + \alpha \ell) < 0.
\]

The only solution is \( a = b = 0 \), that is, \( u = 0 \), so \( N(L_R) \) is trivial.

### 5.2 Introduction to the spectral method; eigenfunctions

1. If \( n \neq m \), then

\[
(u_n, u_m) = \int_0^\ell \sin \left( \frac{n\pi x}{\ell} \right) \sin \left( \frac{m\pi x}{\ell} \right) \, dx
\]

\[
= \frac{1}{2} \int_0^\ell \left\{ \cos \left( \frac{(n-m)\pi x}{\ell} \right) - \cos \left( \frac{(n+m)\pi x}{\ell} \right) \right\} \, dx
\]

\[
= \frac{1}{2} \left[ \frac{\ell}{(n-m)\pi} \sin \left( \frac{(n-m)\pi x}{\ell} \right) - \frac{\ell}{(n+m)\pi} \sin \left( \frac{(n+m)\pi x}{\ell} \right) \right]_0^\ell.
\]

This last expression simplifies to four terms, each including the sine function evaluated at an integer multiple of \( \pi \), and hence each equal to zero. Thus \( (u_n, u_m) = 0 \) for \( n \neq m \).

3. The eigenpairs are

\[
\frac{(2n-1)^2\pi^2}{4\ell^2}, \ \cos \left( \frac{(2n-1)\pi x}{2\ell} \right), \ n = 1, 2, 3, \ldots
\]
The characteristic polynomial is \( r^2 - r + (\lambda - 5) \), so the characteristic roots are

\[
\begin{align*}
  r_1 &= \frac{1 - \sqrt{21 - 4\lambda}}{2}, \\
  r_2 &= \frac{1 + \sqrt{21 - 4\lambda}}{2}. 
\end{align*}
\]  

(5.2)

Case 1: \( \lambda = \frac{21}{4} \). In this case, the characteristic roots are \( r = \frac{1}{2}, \frac{1}{2} \) and the general solution of the ODE is

\[
  u(x) = c_1 e^{x/2} + c_2 xe^{x/2}. 
\]

The boundary condition \( u(0) = 0 \) yields \( c_1 = 0 \), and then the boundary condition \( u(1) = 1 \) implies that \( c_2 = 0 \). Thus there is no nonzero solution to the BVP, and \( \lambda = \frac{21}{4} \) is not an eigenvalue.

Case 2: \( \lambda < \frac{21}{4} \). In this case, the characteristic roots, given by (5.2), are real and distinct. The general solution of the ODE is

\[
  u(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}. 
\]

The boundary conditions lead to the system

\[
\begin{align*}
  c_1 + c_2 &= 0, \\
  c_1 r_1 e^{r_1} + c_2 r_2 e^{r_2} &= 0,
\end{align*}
\]

This system has the unique solution \( c_1 = c_2 = 0 \), so there is no nonzero solution for \( \lambda < \frac{21}{4} \). Hence no \( \lambda < \frac{21}{4} \) is an eigenvalue.

Case 3: \( \lambda > \frac{21}{4} \). In this case, the roots are complex conjugate:

\[
\begin{align*}
  r_1 &= \frac{1}{2} - \theta i, \\
  r_2 &= \frac{1}{2} + \theta i, \\
  \theta &= \sqrt{\frac{4\lambda - 21}{2}}.
\end{align*}
\]

The general solution of the ODE is

\[
  u(x) = (c_1 \cos (\theta x) + c_2 \sin (\theta x)) e^{x/2}. 
\]

The boundary condition \( u(0) = 0 \) yields \( c_1 = 0 \), so any eigenfunction is of the form

\[
  u(x) = \sin (\theta x) e^{x/2}. 
\]

The Neumann condition at \( x = 1 \) is equivalent to the equation

\[
  \tan (\theta) = -2\theta, \quad \theta > 0. 
\]

Although this equation cannot be solved explicitly, a simple graph shows that there are infinitely many solutions \( 0 < \theta_1 < \theta_2 < \cdots \), with

\[
  \theta_k \in \left( \frac{2k - 1}{2}, \frac{2k + 1}{2} \pi \right), \quad k = 1, 2, \ldots. 
\]

Define \( \lambda_k \) by

\[
  \theta_k = \frac{\sqrt{4\lambda_k - 21}}{2}, 
\]

that is,

\[
  \lambda_k = \theta_k^2 + \frac{21}{4}, \quad k = 1, 2, \ldots. 
\]

Then \( \lambda_k, k = 1, 2, \ldots \), are eigenvalues, and the corresponding eigenfunctions are

\[
  v_k(x) = \sin (\theta_k x) e^{x/2}. 
\]

Using Newton’s method, we find that

\[
\begin{align*}
  \theta_1 &\approx 1.8366, \quad \theta_2 \approx 4.8158, \\
  \lambda_1 &\approx 8.6231, \quad \lambda_2 \approx 28.4423.
\end{align*}
\]

The eigenfunctions are \( v_1, v_2 \), as given by the above formula.

A direct calculation shows that

\[
  (v_1, v_2) \approx -0.2092, 
\]

so \( v_1 \) and \( v_2 \) are not orthogonal.
5.3. SOLVING THE BVP USING FOURIER SERIES

7. (a) \[ \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin (n\pi x) \]
   (b) \[ \sum_{n=1}^{\infty} \frac{4\sin \left( \frac{n\pi}{2} \right)}{n^2 \pi^2} \sin (n\pi x) \]
   (c) \[ \sum_{n=1}^{\infty} \frac{12(-1)^{n+1}}{n^3 \pi^3} \sin (n\pi x) \]
   (d) \[ \sum_{n=1}^{\infty} \frac{720(-1)^{n+1}}{n^5 \pi^5} \sin (n\pi x) \]

The errors in approximating the original functions using 10 terms of the Fourier sine series are graphed in Figure 5.1.

9. \( \sin (3\pi x) \) (That is, all of the Fourier sine coefficients are zero, except the third, which is one.)

11. The series have the form

\[ \sum_{n=1}^{\infty} a_n \cos \left( \frac{(2n-1)\pi x}{2} \right), \]

where

(a) \[ a_n = -\frac{4(2+(-1)^n)(2n-1)\pi}{n^3(2n-1)^3} \]
(b) \[ a_n = \frac{32\cos((2n-1)\pi/4)\sin^2((2n-1)\pi/8)}{\pi^2(2n-1)^2} \]
(c) \[ a_n = -\frac{8(24+12(-1)^n)(2n-1)\pi(2n-1)^2\pi^2}{\pi^4(2n-1)^4} \]
(d) \[ a_n = -\frac{8(5760+2880(-1)^n)(2n-1)\pi + 240(2n-1)^2\pi^2 + 7(2n-1)^4\pi^4)}{\pi^6(2n-1)^6} \]

The errors in approximating the original functions using 10 terms of the Fourier quarter-wave cosine series are graphed in Figure 5.2.

5.3 Solving the BVP using Fourier series

1. (a) \[ \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^3 \pi^3} \sin (n\pi x) \]
   (b) \[ x + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^3 \pi^3} \sin (n\pi x) \]

3. (a) \[ \sum_{n=1}^{\infty} \frac{32(-1)^{n+1}}{(2n-1)^4 \pi^4} \sin \left( \frac{(2n-1)\pi x}{2} \right) \]
   (b) \[ 1 + \sum_{n=1}^{\infty} \frac{16((-1)^n \pi + 2(-1)^{n+1} \pi - 2)}{(2n-1)^3 \pi^3} \cos \left( \frac{(2n-1)\pi x}{2} \right) \]
   (c) \[ \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^3 \pi^3} \sin (n\pi x) \]
(d) \[ x + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi(1+n^2\pi^2)} \sin(n\pi x) \]

5. We have

\[ \frac{d^2u}{dx^2}(x) = -x, \quad 0 < x < 1, \]

so

\[
\begin{align*}
\frac{du}{dx}(x) &= \frac{du}{dx}(0) + \int_0^x \frac{d^2u}{dx^2}(s) \, ds \\
&= \frac{du}{dx}(0) - \int_0^x s \, ds \\
&= \frac{du}{dx}(0) - \frac{x^2}{2}.
\end{align*}
\]

Write \( C \) for \( du/dx(0) \); then another integration yields

\[
\begin{align*}
u(x) &= u(0) + \int_0^x \frac{du}{dx}(s) \, ds \\
&= 0 + \int_0^x \left\{ C - \frac{s^3}{2} \right\} \, ds \\
&= Cx - \frac{x^3}{6}.
\end{align*}
\]

Finally, we choose \( C \) so that \( u(1) = 0 \), which yields \( C = 1/6 \). This yields

\[ u(x) = \frac{1}{6} (x - x^3). \]

7. The Fourier quarter-wave sine coefficients of \( -Td^2u/dx^2 \) are

\[ b_n = -\frac{2T}{\ell} \int_0^\ell \frac{d^2u}{dx^2}(x) \sin\left(\frac{(2n-1)\pi x}{2\ell}\right) \, dx, \]

while those of \( u \) are

\[ a_n = \frac{2}{\ell} \int_0^\ell u(x) \sin\left(\frac{(2n-1)\pi x}{2\ell}\right) \, dx. \]
Using integration by parts twice, almost exactly as in (5.19), we can express $b_n$ in terms of $a_n$: 

\[
-\frac{2T}{\ell} \int_0^\ell \frac{d^2 u}{dx^2}(x) \sin \left(\frac{(2n-1)\pi x}{2\ell}\right) dx = -\frac{2T}{\ell} \left\{ \left[ \frac{du}{dx}(x) \sin \left(\frac{(2n-1)\pi x}{2\ell}\right) \right]_0^\ell - \frac{(2n-1)\pi}{2\ell} \int_0^\ell \frac{du}{dx}(x) \cos \left(\frac{(2n-1)\pi x}{2\ell}\right) dx \right\}
\]

\[
= 2T \frac{(2n-1)\pi}{2\ell} \int_0^\ell \frac{du}{dx}(x) \cos \left(\frac{(2n-1)\pi x}{2\ell}\right) dx
\]

(since $\sin(0) = du/dx(\ell) = 0$)

\[
= 2T \frac{(2n-1)\pi}{2\ell} \left\{ \left[ u(x) \cos \left(\frac{(2n-1)\pi x}{2\ell}\right) \right]_0^\ell + \frac{(2n-1)\pi}{2\ell} \int_0^\ell u(x) \sin \left(\frac{(2n-1)\pi x}{2\ell}\right) dx \right\}
\]

(since $u(0) = \cos ((2n-1)\pi/2) = 0$)

\[
= \frac{T(2n-1)^2\pi^2}{4\ell^2} \frac{2}{\ell} \int_0^\ell u(x) \sin \left(\frac{(2n-1)\pi x}{2\ell}\right) dx
\]

\[
= \frac{T(2n-1)^2\pi^2}{4\ell^2} a_n.
\]

This gives the desired result. (Actually, since it is known that the negative second derivative operator is symmetric under the mixed boundary conditions, we can just appeal to (5.25), which is the above calculation written abstractly.)

9. Let $a_1, a_2, a_3, \ldots$ be the Fourier quarter-wave sine coefficients of $u$; then

\[-\kappa \frac{d^2 u}{dx^2}(x) = \sum_{n=1}^{\infty} \frac{\kappa(2n-1)^2\pi^2}{200^2} a_n \sin \left(\frac{(2n-1)\pi x}{200}\right),\]

where $\kappa = 3/2$. We also have

\[
0.001 = \sum_{n=1}^{\infty} \frac{0.004}{(2n-1)\pi} \sin \left(\frac{(2n-1)\pi x}{200}\right).
\]

Setting the two series equal and solving for $a_n$, we find

\[
u(x) = \sum_{n=1}^{\infty} \frac{3.2 \cdot 10^2}{3(2n-1)^3\pi^3} \sin \left(\frac{(2n-1)\pi x}{200}\right).
\]

The temperature distribution is graphed in Figure 5.3.

Figure 5.3: The temperature distribution (in degrees Celsius) in Exercise 5.3.9.
5.4 Finite element methods for BVPs

1. Suppose $f, g \in C_D^2[0, \ell]$, so that
   \[ f(0) = f(\ell) = g(0) = g(\ell) = 0. \]
   Then
   \[
   \left(- \frac{d}{dx} \left(k(x) \frac{df}{dx}\right), g \right) = - \int_0^\ell \frac{d}{dx} k(x) \frac{df}{dx}(x) g(x) \, dx
   = - k(x) \frac{df}{dx}(x) g(x) \bigg|_0^\ell + \int_0^\ell k(x) \frac{df}{dx}(x) \frac{dg}{dx}(x) \, dx \quad \text{(integration by parts)}
   = \int_0^\ell k(x) \frac{df}{dx}(x) \frac{dg}{dx}(x) \, dx \quad \text{(using } g(0) = g(\ell) = 0 \text{)}
   = k(x) f(x) \frac{dg}{dx}(x) \bigg|_0^\ell - \int_0^\ell f(x) \frac{d}{dx} k(x) \frac{dg}{dx}(x) \, dx \quad \text{(integration by parts)}
   = - \int_0^\ell f(x) \frac{d}{dx} \left(k(x) \frac{dg}{dx}(x) \right) \, dx \quad \text{(using } f(0) = f(\ell) = 0 \text{)}
   = \left(f, - \frac{d}{dx} \left(k(x) \frac{dg}{dx}\right) \right).
   \]

3. Take $p(x) = (x - c)^3(d - x)^3$ and $v_{[c, d]}$ as suggested in the hint.

5. Let $v \in C_D^2[0, \ell]$ be a test function. We have
   \[
   - \frac{d}{dx} \left(k(x) \frac{du}{dx}(x)\right) + p(x) u(x) = f(x), \quad 0 < x < \ell
   \Rightarrow - \frac{d}{dx} \left(k(x) \frac{du}{dx}(x)\right) v(x) + p(x) u(x) v(x) = f(x) v(x), \quad 0 < x < \ell
   \Rightarrow - \int_0^\ell \frac{d}{dx} \left(k(x) \frac{du}{dx}(x)\right) v(x) \, dx + \int_0^\ell p(x) u(x) v(x) \, dx = \int_0^\ell f(x) v(x) \, dx
   \Rightarrow -k(x) \frac{du}{dx}(x) v(x) \bigg|_0^\ell + \int_0^\ell k(x) \frac{du}{dx}(x) \frac{dv}{dx}(x) \, dx + \int_0^\ell p(x) u(x) v(x) \, dx = \int_0^\ell f(x) v(x) \, dx
   \Rightarrow \int_0^\ell k(x) \frac{du}{dx}(x) \frac{dv}{dx}(x) \, dx + \int_0^\ell p(x) u(x) v(x) \, dx = \int_0^\ell f(x) v(x) \, dx
   \Rightarrow \int_0^\ell \left\{k(x) \frac{du}{dx}(x) \left(\frac{dv}{dx}(x) + p(x) u(x) v(x)\right)\right\} \, dx = \int_0^\ell f(x) v(x) \, dx.
   \]
   The last step follows from the fact that $v(0) = v(\ell) = 0$. Thus the weak form of the given BVP is
   \[
   \text{find } u \in C_D^2[a, b] \text{ such that } \int_0^\ell \left\{k(x) \frac{du}{dx}(x) \left(\frac{dv}{dx}(x) + p(x) u(x) v(x)\right)\right\} \, dx = \int_0^\ell f(x) v(x) \, dx \text{ for all } v \in C_D^2[a, b].
   \]

7. The calculation is the same as in Exercise 5; we integrate by parts only in the first integral, and obtain the following weak form: Find $u \in C_D^2[a, b]$ such that
   \[
   \int_0^\ell \left\{k(x) \frac{du}{dx}(x) \frac{dv}{dx}(x) + c(x) \frac{du}{dx}(x) v(x) + p(x) u(x) v(x)\right\} \, dx = \int_0^\ell f(x) v(x) \, dx \text{ for all } v \in C_D^2[a, b].
   \]
   Notice that now the left-hand side is not symmetric in $u$ and $v$.

9. Repeating the calculation beginning on page 167, we obtain
   \[
   \frac{\partial}{\partial t} \left[ \frac{1}{2} \int_0^\ell Ap(x) \left(\frac{\partial u}{\partial t}\right)^2 \, dx + \frac{1}{2} \int_0^\ell Ak(x) \left(\frac{\partial u}{\partial x}\right)^2 \, dx \right]
   = - \int_0^\ell c \left(\frac{\partial u}{\partial t}\right)^2 \, dx, \quad t > 0.
   \]
This shows that derivative with respect to time of the total energy is always negative, and hence that the total energy is always decreasing.

5.5 The Galerkin method

1. (a) Yes.
   (b) No, \( a(f, f) = 0 \) for \( f(x) = 1 \), but \( f \neq 0 \).
   (c) No, \( a(f, f) = 0 \) for \( f(x) = 1 \), but \( f \neq 0 \).

3. Let \( F_N = \text{span} \{ \sin (\pi x), \sin (2\pi x), \ldots, \sin (N\pi x) \} \).

We will apply the Galerkin method to the BVP
\[
-ku''(x) = f(x), \quad 0 < x < \ell, \\
u(0) = 0, \\
u(\ell) = 0,
\]
using \( F_N \) as the approximating subspace. The weak form of the BVP is
\[
\text{find } u \in C^2_0[a, b] \text{ such that } \int_0^\ell k u''(x) v''(x) \, dx = \int_0^\ell f(x)v(x) \, dx \text{ for all } v \in C^2_0[a, b],
\]
and the Galerkin method takes the form
\[
\text{find } v_N = \sum_{j=1}^N u_j \sin (j\pi x) \text{ such that } \int_0^\ell k u''(x) v''(x) \, dx = \int_0^\ell f(x)v(x) \, dx \text{ for all } v \in F_N.
\]

We find the coefficients \( u_j, j = 1, 2, \ldots, N \), by solving \( Ku = f \), where
\[
K_{ij} = \int_0^\ell k \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial x} \, dx, \quad F_i = \int_0^\ell f(x)\phi_i(x) \, dx
\]
and \( \phi_j(x) = \sin (j\pi x) \). We have
\[
\int_0^\ell k \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial x} \, dx = \int_0^\ell k ij\pi^2 \cos (j\pi x) \cos (i\pi x) \, dx = \begin{cases} \frac{k j^2 \pi^2}{2}, & i = j, \\ 0, & i \neq j. \end{cases}
\]

This last step follows from the fact that the functions \( \cos (\pi x), \cos (2\pi x), \ldots, \cos (N\pi x) \) are mutually orthogonal with respect to the \( L^2(0,1) \) inner product. Also, notice that
\[
\int_0^\ell f(x)\phi_j(x) \, dx = \int_0^\ell f(x)\sin (j\pi x) \, dx = \frac{c_j}{2}
\]
where \( c_j \) is the Fourier sine coefficient of \( f \). Since \( K \) is diagonal, it is easy to solve \( Ku = f \):
\[
u_j = \frac{c_j/2}{kj^2\pi^2/2} = \frac{c_j}{kj^2\pi^2}.
\]

Thus
\[
v_N(x) = \sum_{j=1}^N \frac{c_j}{kj^2\pi^2} \sin (j\pi x).
\]

This is exactly the solution obtained by the method of Fourier series in Section 5.3.2.

5. (a) The set \( S \) is the span of \( \{x, x^2\} \) and therefore is a subspace.
   (b) As discussed in the text, \( a(\cdot, \cdot) \) automatically satisfies two of the properties of an inner product. Every function \( p \) in \( S \) satisfies \( p(0) = 0 \), so \( a(\cdot, \cdot) \) also satisfies the third property (\( a(u, u) = 0 \) implies that \( u = 0 \) for exactly the same reason as given in the text for the subspace \( V \) (see page 174).
   (c) The best approximation is \( p(x) = (9 - 3e)x^2 + (4e - 10)x \).
(d) The bilinear form is not an inner product on \( P_2 \), since \( a(1, 1) = 0 \) but \( 1 \neq 0 \) (a constant is “invisible” to the energy inner product and norm). Therefore, every polynomial of the form \((9 - 3e)x^2 + (4e - 10)x + C \in S\) is equidistant from \( f(x) = e^x \) in the energy norm.

7. Write \( p(x) = x(1 - x) \) and \( q(x) = x(1/2 - x)(1 - x) \). The approximation will be \( v(x) = u_1p(x) + u_2q(x) \), where \( Ku = f \). The stiffness matrix \( K \) is

\[
K = \begin{bmatrix}
\int_0^1 (1 + x) \frac{d}{dx} p(x) \frac{d}{dx} q(x) dx & \int_0^1 (1 + x) \frac{d}{dx} p(x) \frac{d}{dx} q(x) dx \\
\int_0^1 (1 + x) \frac{d}{dx} q(x) \frac{d}{dx} p(x) dx & \int_0^1 (1 + x) \frac{d}{dx} q(x) \frac{d}{dx} q(x) dx
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{1}{2} & -\frac{1}{30} \\
-\frac{1}{30} & \frac{1}{40}
\end{bmatrix},
\]

and the load vector \( f \) is

\[
f = \begin{bmatrix}
\int_0^1 xp(x) dx \\
\int_0^1 xq(x) dx
\end{bmatrix} = \begin{bmatrix}
\frac{1}{12} \\
-\frac{1}{120}
\end{bmatrix}.
\]

Therefore,

\[
u = \begin{bmatrix}
0.16412 \\
-0.038168
\end{bmatrix}.
\]

The exact solution is

\[
u(x) = -\frac{1}{4}x^2 + \frac{1}{2}x - \frac{1}{4\ln 2} \ln (1 + x).
\]

The exact and approximate solutions are graphed in Figure 5.4.

![Figure 5.4: The exact and approximate solutions from Exercise 5.5.7.](image)

9. (a) It will take about \( 10^3 \) times as long, or 1000 seconds (almost 17 minutes), to solve a \( 1000 \times 1000 \) system, and \( 10^6 \) times as long, or \( 10^6 \) seconds (about 11.5 days) to solve a \( 10000 \times 10000 \) system.

(b) Gaussian elimination consists of a forward phase, in which the diagonal entries are used to eliminate nonzero entries below and in the same column, and a backward phase, in which the diagonal entries are used to eliminate nonzero entries above and in the same column. During the forward phase, at a typical step, there is only 1 nonzero entry below the diagonal, and only 5 arithmetic operations are required to eliminate it (1 division to compute the multiplier, and 2 multiplications and 2 additions to add a multiple of the current row to the next). Thus the forward phase requires \( O(5n) \) operations. A typical step of the backward phase requires 3 arithmetic operations (a multiplication and an addition to adjust the right-hand side and a division to solve for the unknown). Thus the backward phase requires \( O(3n) \) operations. The grand total is \( O(8n) \) operations.

(c) It will take about 10 times as long, or 0.1 seconds, to solve a \( 1000 \times 1000 \) tridiagonal system, and 100 times as long, or 1 second, to solve a \( 10000 \times 10000 \) tridiagonal system.
Piecewise polynomials and the finite element method

1. We wish to compute the piecewise linear finite element approximation to the solution of
\[-d^2u/dx^2 = x, \quad 0 < x < 1,\]
\[u(0) = 0,\]
\[u(1) = 0\]

using a uniform mesh with four elements. Such a mesh corresponds to \( h = 1/4 \), and there are three basis functions. The stiffness matrix is
\[
K = \begin{bmatrix}
8 & -4 & 0 \\
-4 & 8 & -4 \\
0 & -4 & 8
\end{bmatrix}
\]
(as computed in Example 5.18 in the text). The load vector \( F \in \mathbb{R}^3 \) is defined by
\[
F_i = \int_0^1 f(x) \phi_i(x) dx.
\]
A direct calculation shows that \( F_1 = 1/16, F_2 = 1/8, F_3 = 3/16 \). Solving \( Ku = f \) yields
\[
u = \begin{bmatrix}
\frac{5}{128} \\
\frac{1}{16} \\
\frac{7}{128}
\end{bmatrix}.
\]

The exact solution to the BVP is \( u(x) = (x - x^3)/6 \). Figure 5.5 shows the exact solution and the piecewise linear approximation.

![Figure 5.5: The exact (dashed curve) and approximate (solid curve) solutions from Exercise 5.6.1.](image)

3. (b) Here are the errors for \( n = 10, 20, 40, 80 \):

<table>
<thead>
<tr>
<th>( n )</th>
<th>maximum error</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>( 1.6586 \times 10^{-3} )</td>
</tr>
<tr>
<td>20</td>
<td>( 4.3226 \times 10^{-4} )</td>
</tr>
<tr>
<td>40</td>
<td>( 1.1036 \times 10^{-4} )</td>
</tr>
<tr>
<td>80</td>
<td>( 2.7881 \times 10^{-5} )</td>
</tr>
</tbody>
</table>

We see that, when \( n \) is doubled, the error decreases by a factor of approximately four. Thus
\[
\text{error} = O \left( \frac{1}{n^2} \right).
\]

5. The exact solution is \( u(x) = x(1 - x^3)/12 \). Here are the errors for \( n = 10, 20, 40, 80 \):
CHAPTER 5. BOUNDARY VALUE PROBLEMS IN STATICS

We see that, when \( n \) is doubled, the error decreases by a factor of approximately four. Thus

\[
\text{error} = O \left( \frac{1}{n^2} \right).
\]

The weak form is

\[
\text{find } u \in V \text{ such that } \int_0^\ell \left\{ k(x) \frac{du}{dx}(x) \frac{dv}{dx}(x) + p(x)u(x)v(x) \right\} dx = \int_0^\ell f(x)v(x) dx \text{ for all } v \in V.
\]

The bilinear form is

\[
a(u, v) = \int_0^\ell \left\{ k(x) \frac{du}{dx}(x) \frac{dv}{dx}(x) + p(x)u(x)v(x) \right\} dx.
\]

We wish to apply the piecewise linear finite element method to the BVP

\[
\begin{aligned}
-\frac{d^2u}{dx^2} + 2u &= \frac{1}{2} - x, \quad 0 < x < 1, \\
u(0) &= 0, \\
u(1) &= 0.
\end{aligned}
\]

We use a sequence of increasingly fine uniform meshes. The weak form of the BVP is

\[
u \in C_D^2[0,1], \quad \int_0^1 \frac{dv_n}{dx}(x) \frac{dv}{dx}(x) dx + \int_0^1 2u(x)v(x) dx = \int_0^1 \left( \frac{1}{2} - x \right) v(x) dx \text{ for all } v \in C_D^2[a,b],
\]

and the Galerkin method requires the solution of

\[
v_n \in V_n, \quad \int_0^1 \frac{dv_n}{dx}(x) \frac{dv_n}{dx}(x) dx + 2 \int_0^1 v_n(x)v(x) dx = \int_0^1 \left( \frac{1}{2} - x \right) v(x) dx \text{ for all } v \in V_n.
\]

The second integral on the left side of the variational equation leads to the matrix \( M \in \mathbb{R}^{(n-1)\times(n-1)} \), where

\[
M_{ij} = \int_0^1 \phi_j(x)\phi_i(x) dx
\]

and \( \phi_j \) is the \( j \)th standard basis function for the space of continuous piecewise linear functions. The matrix \( M \) can be computed explicitly (similarly to how \( K \) was computed in Example 5.18 in the text), and the result is

\[
M_{ij} = \begin{cases} 
\frac{2h}{3}, & j = i - 1 \text{ or } j = i + 1, \\
-\frac{1}{2h}, & j = i, \\
0, & \text{otherwise}.
\end{cases}
\]

We then must solve \((K + 2M)u = f\), where \( K \) is the usual stiffness matrix (as in Example 5.18) and the load vector \( f \) is defined by

\[
f_i = \int_0^1 \left( \frac{1}{2} - x \right) \phi_i(x) dx = \frac{h}{2}(1 - 2h), \quad i = 1, 2, \ldots, n - 1.
\]

After solving \((K + 2M)u = f\) to get the piecewise linear approximation for each \( n \), we estimate the maximum error by evaluating both the exact function and the approximation on a finer mesh. The results are shown in the following table.

<table>
<thead>
<tr>
<th>( n )</th>
<th>\text{error}</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>5.4953 \cdot 10^{-4}</td>
</tr>
<tr>
<td>20</td>
<td>1.4660 \cdot 10^{-4}</td>
</tr>
<tr>
<td>40</td>
<td>3.7842 \cdot 10^{-5}</td>
</tr>
<tr>
<td>80</td>
<td>9.6121 \cdot 10^{-6}</td>
</tr>
<tr>
<td>160</td>
<td>2.4222 \cdot 10^{-6}</td>
</tr>
<tr>
<td>320</td>
<td>6.0794 \cdot 10^{-7}</td>
</tr>
</tbody>
</table>
These results show that when \( n \) is doubled, the error is reduced by a factor of approximately 4. Thus it appears that the error is \( O(1/n^2) \).
Chapter 6

Heat Flow and Diffusion

6.1 Fourier series methods for the heat equation

1. The steady-state solution of Example 6.2 satisfies the BVP

\[-D \frac{d^2 u}{dx^2} = 10^{-7}, \ 0 < x < 100,\]
\[u(0) = 0,\]
\[u(100) = 0.\]

Either by solving this BVP or by taking the limit (as \(t \to \infty\)) of the solution of Example 6.2, we find that the steady-state solution is

\[u_s(x) = \sum_{n=1}^{\infty} \frac{2 \cdot 10^{-3} (1 - (-1)^n)}{Dn^3 \pi^3} \sin \left( \frac{n \pi x}{100} \right).\]

3. The solution is

\[u(x,t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n \pi} e^{-n^2 \pi^2 t} \sin (n \pi x).\]

A graph of \(u(\cdot,0.1)\) is given in Figure 6.1.

![Graph of u(x,t) for Exercise 6.1.3](image)

Figure 6.1: The snapshot \(u(\cdot,0.1)\), together with the initial temperature distribution, for Exercise 6.1.3.

5. With

\[p(x,t) = g(t) + \frac{x}{\ell} (h(t) - g(t))\]

and \(v(x,t) = u(x,t) - p(x,t)\), we have

\[\rho c \frac{\partial v}{\partial t} - \kappa \frac{\partial^2 v}{\partial x^2} = \rho c \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} - \left( \rho c \frac{\partial p}{\partial t} - \kappa \frac{\partial^2 p}{\partial x^2} \right) = f(x,t) - \rho c \left( \frac{dg}{dt}(t) + \frac{x}{\ell} \left( \frac{dh}{dt}(t) - \frac{dg}{dt}(t) \right) \right).\]
We also have
\[
v(x, t_0) = u(x, t_0) - p(x, t_0) = \psi(x) - g(t_0) - \frac{x}{\ell} (h(t_0) - g(t_0))
\]
and
\[
\begin{align*}
v(0, t) &= u(0, t) - p(0, t) = g(t) - g(t) = 0, \\
v(\ell, t) &= u(\ell, t) - p(\ell, t) = h(t) - h(t) = 0.
\end{align*}
\]
We define
\[
g(x, t) = f(x, t) - \rho c \left( \frac{dg}{dt}(t) + \frac{x}{\ell} \left( \frac{dh}{dt}(t) - \frac{dg}{dt}(t) \right) \right)
\]
and
\[
\phi(x) = \psi(x) - g(t_0) - \frac{x}{\ell} (h(t_0) - g(t_0)).
\]
Then \(v\) satisfies
\[
\rho c \frac{\partial v}{\partial t} - \kappa \frac{\partial^2 v}{\partial x^2} = g(x, t), \quad 0 < x < \ell, \quad t > 0,
\]
\[
v(x, t_0) = \phi(x), \quad 0 < x < \ell,
\]
\[
v(0, t) = 0, \quad t > t_0,
\]
\[
v(\ell, t) = 0, \quad t > t_0.
\]

7. Define \(v(x, t) = u(x, t) - x \cos(t)\). Then \(v\) satisfies the following IBVP (with homogeneous boundary conditions, but with a nonzero source term):
\[
\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = x \sin(t), \quad 0 < x < 1, \quad t > 0,
\]
\[
v(x, 0) = 0, \quad 0 < x < 1,
\]
\[
v(0, t) = 0, \quad t > 0,
\]
\[
v(1, t) = 0, \quad t > 0.
\]
This IBVP has solution
\[
v(x, t) = \sum_{n=1}^{\infty} a_n(t) \sin(n \pi x),
\]
where
\[
a_n(t) = \frac{2(-1)^n}{n \pi (n^4 \pi^4 + 1)} \left( \cos(t) - e^{-n^2 \pi^2 t} - n^2 \pi^2 \sin(t) \right).
\]
The solution to the original IBVP is then \(u(x, t) = v(x, t) + x \cos(t)\). A graph of \(u(\cdot, 1.0)\) is given in Figure 6.2.

Figure 6.2: The snapshot \(u(\cdot, 1.0)\), together with the initial temperature distribution, for Exercise 6.1.7.

9. The temperature \(u(x, t)\) satisfies the IBVP
\[
\rho c \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < 100, \quad t > 0,
\]
\[
u(x, 0) = 5, \quad 0 < x < 100,
\]
\[
u(0, t) = 0, \quad t > 0,
\]
\[
u(100, t) = 0, \quad t > 0.
\]
The solution is
\[ u(x,t) = \sum_{n=1}^{\infty} \frac{10(1 - (-1)^n)}{n\pi} e^{-\frac{n^2\pi^2}{10000\rho_c} t} \sin \left( \frac{n\pi x}{100} \right). \]

The temperature at the midpoint after 20 minutes is
\[ u(50,1200) \approx 1.58 \text{ degrees Celsius}. \]

11. (a) The steady-state temperature is
\[ u_s(x) = \frac{x}{20}. \]

(b) The temperature \( u(x,t) \) satisfies the IBVP
\[
\rho c \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < 100, \quad t > 0,
\]
\[
u(x,0) = 5, \quad 0 < x < 100,
\]
\[ u(0,t) = 0, \quad t > 0,
\]
\[ u(100,t) = 5, \quad t > 0.
\]

The solution is
\[ u(x,t) = \frac{x}{20} + \sum_{n=1}^{\infty} \frac{10}{n\pi} e^{-\frac{n^2\pi^2}{10000\rho_c} t} \sin \left( \frac{n\pi x}{100} \right). \]

Considering the results of Exercise 6.1.9, it is obvious that at least several thousand seconds will elapse before the temperature is within 1% of steady state, so we can accurately estimate \( u(x,t) \) using a single term of the Fourier series:
\[ u(x,t) \approx \frac{x}{20} + 10 \pi e^{-\frac{\pi^2}{10000\rho_c} t} \sin \left( \frac{\pi x}{100} \right). \]

We want to find \( t \) large enough that
\[ \frac{|u(x,t) - u_s(x)|}{|u_s(x)|} \leq 0.01 \]
for all \( x \in [0,100] \). Using our approximation for \( u \), this is equivalent to
\[ \left| \frac{200 \sin \left( \frac{\pi x}{100} \right)}{\pi x} \right| e^{-\frac{\pi^2}{10000\rho_c} t} \leq 0.01. \]

A graph shows that
\[ \left| \frac{200 \sin \left( \frac{\pi x}{100} \right)}{\pi x} \right| \leq 2 \]
for all \( x \), so we need
\[ e^{-\frac{\pi^2}{10000\rho_c} t} \leq 0.005. \]

This yields
\[ t \geq -\frac{\pi^2}{100^2 \rho_c \ln 0.005} \frac{\kappa}{\pi^2} \approx 4520. \]

About 75 minutes and 20 seconds are required.

6.2 Pure Neumann conditions and the Fourier cosine series

1. The solution is
\[ u(x,t) = d_0 + \sum_{n=1}^{\infty} d_n e^{-n^2\pi^2 t} \cos (n\pi x), \]
where
\[ d_0 = \int_0^1 x(1-x) \, dx = \frac{1}{6}, \quad d_n = 2 \int_0^1 x(1-x) \cos (n\pi x) \, dx = \frac{2((-1)^{n+1}-1)}{n^2\pi^2}. \]

The graphs are given in Figure 6.3.
Figure 6.3: The solution $u(x, t)$ from Exercise 6.2.4 at times 0, 0.02, 0.04, and 0.06, along with the steady-state solution. These solutions were estimated using 10 terms in the Fourier series.

3. (a) If $u, v \in C^2_N[0, \ell]$, then

\[
(L_N u, v) = -\int_0^\ell \frac{d^2 u}{dx^2}(x)v(x)\,dx
= -\left. \frac{d u}{dx}(x)v(x) \right|_0^\ell + \int_0^\ell \frac{d u}{dx}(x)\frac{d v}{dx}(x)\,dx
= \int_0^\ell \frac{d u}{dx}(x)\frac{d v}{dx}(x)\,dx \quad \text{(since $\frac{d a}{dx}(0) = \frac{d a}{dx}(\ell) = 0$)}
= u(x)\left. \frac{d v}{dx}(x) \right|_0^\ell - \int_0^\ell u(x)\frac{d^2 v}{dx^2}(x)\,dx
= -\int_0^\ell u(x)\frac{d^2 v}{dx^2}(x)\,dx \quad \text{(since $\frac{d a}{dx}(0) = \frac{d a}{dx}(\ell) = 0$)}
= (u, L_N v).
\]

This shows that $L_N$ is symmetric.

(b) Suppose $\lambda$ is an eigenvalue of $L_N$ and $u$ is a corresponding eigenvector, normalized so that $(u, u) = 1$. Then

\[
\lambda = \lambda(u, u) = (\lambda u, u) = (L_N u, u) = -\int_0^\ell \frac{d^2 u}{dx^2}(x)u(x)\,dx
= -\left. \frac{d u}{dx}(x)u(x) \right|_0^\ell + \int_0^\ell \left( \frac{d u}{dx}(x) \right)^2\,dx
= \int_0^\ell \left( \frac{d u}{dx}(x) \right)^2\,dx \quad \text{(since $\frac{d a}{dx}(0) = \frac{d a}{dx}(\ell) = 0$)}
\geq 0.
\]

Thus $L_N$ cannot have any negative eigenvalues.

5. (a) Suppose $u$ is a solution to the BVP. Then

\[
\int_0^\ell f(x)\,dx = -\kappa \int_0^\ell \frac{d^2 u}{dx^2}(x)\,dx = \kappa \left( \frac{d u}{dx}(0) - \frac{d u}{dx}(\ell) \right) = \kappa(a - b).
\]

This is the compatibility condition:

\[
\int_0^\ell f(x)\,dx = \kappa(a - b).
\]

(b) The operator $K : C^2_N[0, \ell] \to C[0, \ell]$ defined by

\[
Ku = -\kappa \frac{d^2 u}{dx^2} + u
\]
has eigenpairs
\[ \lambda_0 = 1, \ \gamma_0(x) = 1, \ \text{and} \ \lambda_n = 1 + \frac{n\pi}{\ell}, \ \gamma_n(x) = \cos \left( \frac{n\pi x}{\ell} \right), \ n = 1, 2, 3, \ldots. \]

The method of Fourier series can be applied to show that a unique solution exists for each \( f \in C[0, \ell] \). (The key is that 0 is not an eigenvalue of \( K \), as it is of \( L_N \).)

7. As shown in this section, the solution to the IBVP is
\[ u(x, t) = d_0 + \sum_{n=1}^{\infty} d_n e^{-n^2 \pi^2 t/\ell^2} \cos \left( \frac{n\pi x}{\ell} \right), \]
where \( d_0, d_1, d_2, \ldots \) are the Fourier cosine coefficients of \( \psi \). We have
\[
\left| \sum_{n=1}^{\infty} d_n e^{-n^2 \pi^2 t/\ell^2} \cos \left( \frac{n\pi x}{\ell} \right) \right| \leq \sum_{n=1}^{\infty} |d_n| e^{-n^2 \pi^2 t/\ell^2} \left| \cos \left( \frac{n\pi x}{\ell} \right) \right|
\leq \sum_{n=1}^{\infty} |d_n| e^{-n^2 \pi^2 t/\ell^2}
\leq e^{-\pi^2 t/\ell^2} \sum_{n=1}^{\infty} |d_n| e^{-(n^2-1)\pi^2 t/\ell^2}.
\]
This last series certainly converges (as can be proved, for example, using the comparison test), and
\[ e^{-\pi^2 t/\ell^2} \to 0 \text{ as } t \to \infty. \]
This shows that
\[ d_0 + \sum_{n=1}^{\infty} d_n e^{-n^2 \pi^2 t/\ell^2} \cos \left( \frac{n\pi x}{\ell} \right) \to d_0 \text{ as } t \to \infty. \]
The limit \( d_0 \) is
\[ \frac{1}{\ell} \int_0^{\ell} \psi(x) \, dx, \]
the average of the initial temperature distribution.

9. The steady-state temperature is about 0.992 degrees Celsius.

11. (a) Suppose that the Fourier sine series of \( u(x, t) \) on \( (0, \ell) \) is
\[ u(x, t) = \sum_{n=1}^{\infty} a_n(t) \sin \left( \frac{n\pi x}{\ell} \right), \ a_n(t) = \frac{2}{\ell} \int_0^{\ell} u(x, t) \sin \left( \frac{n\pi x}{\ell} \right) \, dx \]
and \( u \) satisfies
\[ \frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(\ell, t) = 0 \text{ for all } t > 0. \]
The nth Fourier coefficient of \(-\partial^2 u/\partial x^2\) is computed as follows:
\[
-\frac{2}{\ell} \int_0^{\ell} \frac{\partial^2 u}{\partial x^2}(x, t) \sin \left( \frac{n\pi x}{\ell} \right) \, dx
= -\frac{2}{\ell} \left[ \frac{\partial u}{\partial x}(x, t) \sin \left( \frac{n\pi x}{\ell} \right) \right]_0^\ell - \frac{n\pi}{\ell} \int_0^{\ell} \frac{\partial u}{\partial x}(x, t) \cos \left( \frac{n\pi x}{\ell} \right) \, dx
= \frac{2n\pi}{\ell^2} \int_0^{\ell} \frac{\partial u}{\partial x}(x, t) \cos \left( \frac{n\pi x}{\ell} \right) \, dx
= \frac{2n\pi}{\ell^2} \left[ u(x, t) \cos \left( \frac{n\pi x}{\ell} \right) \right]_0^\ell + \frac{n\pi}{\ell} \int_0^{\ell} u(x, t) \sin \left( \frac{n\pi x}{\ell} \right) \, dx
= \frac{2n\pi}{\ell^2} \left[ u(x, t) - u(0, t) \right] + \frac{n^2 \pi^2}{\ell^2} a_n(t).
\]
Since the values \( u(0, t) \) and \( u(\ell, t) \) are unknown, we see that it is not possible to express the Fourier sine coefficients of \(-\partial^2 u/\partial x^2\) in terms of \( a_1(t), a_2(t), \ldots \).
(b) The Fourier sine series of \( u(x, t) = t \) is
\[
\sum_{n=1}^{\infty} 2 \frac{(1 - (-1)^n)}{n\pi} t \sin (n\pi x),
\]
and the formal calculation of the sine series of \(-\partial^2 u/\partial x^2\) yields
\[
\sum_{n=1}^{\infty} 2 \frac{(1 - (-1)^n)}{n\pi} n\pi t \sin (n\pi x).
\]
However,
\[-\frac{\partial^2 u}{\partial x^2}(x, t) = 0,
\]
and so all of the Fourier sine coefficients of \(-\partial^2 u/\partial x^2\) should be zero. Thus the formal calculation is wrong.

### 6.3 Periodic boundary conditions and the full Fourier series

1. The formula for the solution \( u \) is exactly the same as in Example 6.6, with a different value for \( \kappa \) (4.29 instead of 3.17). This implies that the amplitude of the solution is reduced by about 26%. Therefore, there is less variation in the temperature distribution in the silver ring as opposed to the gold ring.

3. (a) The IBVP is
\[
\rho c \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0, \quad -5\pi < x < 5\pi, \quad t > 0,
\]
\[
u(x, 0) = \psi(x), \quad -5\pi < x < 5\pi,
\]
\[
u(-5\pi, t) = \nu(5\pi, t), \quad t > 0,
\]
\[
\frac{\partial \nu}{\partial x}(-5\pi, t) = \frac{\partial \nu}{\partial x}(5\pi, t), \quad t > 0.
\]
(b) The solution is
\[
u(x, t) = c_0 + \sum_{n=1}^{\infty} c_n e^{-\frac{\kappa n^2}{2\rho c^2} t} \cos \left( \frac{n\pi x}{5} \right),
\]
where
\[
c_0 = 25 + \frac{\pi^4}{9},
\]
\[
c_n = \frac{10(-1)^{n+1}}{n^4}, \quad n = 1, 2, 3, \ldots
\]
(c) The steady-state temperature is the constant \( u_s = 25 + \pi^4/9 \) degrees Celsius.
(d) We must choose \( t \) so that
\[
\frac{|u_s - \nu(x, t)|}{|u_s|} \leq 0.01
\]
holds for every \( x \in [-5\pi, 5\pi] \). By trial and error, we find that about 360 seconds (6 minutes) are required.

5. (a) To show that \( L_p \) is symmetric, we perform the now familiar calculation: we form the integral \( (L_p u, v) \) and integrate by parts to obtain \( (u, L_p v) \). The boundary term from the first integration by parts is
\[
- \left[ \frac{du}{dx} v(x) \right]_{-\ell}^{\ell} = \frac{du}{dx}(\ell)v(-\ell) - \frac{du}{dx}(-\ell)v(\ell).
\]
Since both \( u \) and \( v \) satisfy periodic boundary conditions, we have
\[
v(-\ell) = v(\ell), \quad \frac{du}{dx}(-\ell) = \frac{du}{dx}(\ell),
\]
so the boundary term vanishes. The boundary term from the second integration by parts vanishes for exactly the same reason.
6.3. PERIODIC BOUNDARY CONDITIONS AND THE FULL FOURIER SERIES

(b) Suppose $L_p u = \lambda u$, where $u$ has been normalized: $(u, u) = 1$. Then

$$\lambda = \lambda(u, u) = (\lambda u, u) = (L_p u, u) = - \int_{-\ell}^{\ell} \frac{d^2 u}{dx^2}(x) u(x) \, dx$$

$$= - \left[ \frac{du}{dx}(x) u(x) \right]_{-\ell}^{\ell} + \int_{-\ell}^{\ell} \left( \frac{du}{dx}(x) \right)^2 \, dx$$

$$= \int_{-\ell}^{\ell} \left( \frac{du}{dx}(x) \right)^2 \, dx$$

$$\geq 0.$$ 

The boundary terms vanishes because of the periodic boundary conditions:

$$\frac{du}{dx}(-\ell) u(-\ell) = \frac{du}{dx}(\ell) u(\ell).$$

7. (a) Since the ring is completely insulated, a steady-state temperature distribution cannot exist unless the net amount of heat being added to the ring is zero. This is exactly the same situation as a straight bar with the ends, as well as the sides, insulated.

(b) Suppose $u$ is a solution to (6.21). Then

$$\int_{-\ell}^{\ell} f(x) \, dx = -\kappa \int_{-\ell}^{\ell} \frac{\partial^2 u}{\partial x^2}(x, t) \, dx = -\kappa \left[ \frac{\partial u}{\partial x}(x, t) \right]_{-\ell}^{\ell} = \kappa \left( \frac{\partial u}{\partial x}(-\ell, t) - \frac{\partial u}{\partial x}(\ell, t) \right) = 0.$$

The last step follows from the periodic boundary conditions.

(c) The negative second derivative operator, subject to boundary conditions, has a nontrivial null space, namely, the space of all constant functions on $(-\ell, \ell)$. In analogy to the Fredholm alternative for symmetric matrices, we would expect a solution to the boundary value problem to exist if and only if the right-hand-side function is orthogonal to this null space. This condition is

$$\int_{-\ell}^{\ell} c f(x) \, dx = 0 \text{ for all } c \in \mathbb{R},$$

or simply

$$\int_{-\ell}^{\ell} f(x) \, dx = 0.$$

9. Consider the IBVP

$$\rho c \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} + pu = f(x, t), \quad -\ell < x < \ell, \quad t > t_0,$$

$$u(x, t_0) = \psi(x), \quad -\ell < x < \ell,$$

$$u(-\ell, t) = u(\ell, t), \quad t > t_0,$$

$$\frac{\partial u}{\partial x}(-\ell, t) = \frac{\partial u}{\partial x}(\ell, t), \quad t > t_0.$$ 

Assume that $p$ is a constant.

(a) We wish to derive the solution to the IBVP using the Fourier series method. The calculation is very similar to that carried out in Section 6.3.3 in the text, and we will use the same notation as in that section. We represent the solution $u$ as the series

$$u(x) = a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos \left( \frac{n\pi x}{\ell} \right) + b_n \sin \left( \frac{n\pi x}{\ell} \right) \right\}.$$ 

Substituting this series into the left side of the PDE yields

$$\rho c \frac{\partial u}{\partial t}(x, t) - \kappa \frac{\partial^2 u}{\partial x^2}(x, t) + pu(x, t)$$

$$= \rho c \frac{da_0}{dt} + p a_0(t) + \sum_{n=1}^{\infty} \left\{ \left[ \rho c \frac{da_n}{dt}(t) + \left( \frac{k \pi^2}{\ell^2} + p \right) a_n(t) \right] \cos \left( \frac{n\pi x}{\ell} \right) + \right\}$$

$$\left[ \rho c \frac{db_n}{dt}(t) + \left( \frac{k \pi^2}{\ell^2} + p \right) b_n(t) \right] \sin \left( \frac{n\pi x}{\ell} \right).$$
Following the derivation in Section 6.3.3, we find \( a_0, a_1, \ldots, b_1, b_2, \ldots \) by solving the IVPs

\[
\rho c \frac{da_0}{dt} + pa_0 = c_0(t), \quad a_0(t_0) = p_0, \\
\rho c \frac{da_n}{dt} + \left( \frac{\kappa n^2 \pi^2}{\ell^2} + p \right) a_n = c_n(t), \quad a_n(t_0) = p_n, \quad n = 1, 2, \ldots, \\
\rho c \frac{db_n}{dt} + \left( \frac{\kappa n^2 \pi^2}{\ell^2} + p \right) b_n = d_n(t), \quad b_n(t_0) = q_n, \quad n = 1, 2, \ldots.
\]

(b) The eigenvalues of the spatial operator are \( \lambda_0 = p \) and

\[
\lambda_n = \frac{\kappa n^2 \pi^2}{\ell^2} + p, \quad n = 1, 2, \ldots.
\]

Therefore, \( \lambda_n \geq \lambda_0 \) for all \( n \), and all the eigenvalues are positive if and only if \( p > 0 \).

11. Let \( f : (-\ell, \ell) \to \mathbb{R} \) be an even function. We wish to prove that the full Fourier series of \( f \) reduces to the cosine series of \( f \), regarded as a function on \((0, \ell)\). The full Fourier series of \( f \) is

\[
f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \left( \frac{n\pi x}{\ell} \right) + b_n \sin \left( \frac{n\pi x}{\ell} \right) \right),
\]

where

\[
a_0 = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) \, dx, \\
a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \left( \frac{n\pi x}{\ell} \right) \, dx, \quad n = 1, 2, \ldots, \\
b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \left( \frac{n\pi x}{\ell} \right) \, dx, \quad n = 1, 2, \ldots.
\]

For any function \( g : (-\ell, \ell) \to \mathbb{R} \), if \( g \) is even, then

\[
\int_{-\ell}^{\ell} g(x) \, dx = 2 \int_{0}^{\ell} g(x) \, dx,
\]

while if \( g \) is odd, then

\[
\int_{-\ell}^{\ell} g(x) \, dx = 0.
\]

Since \( f \) is even,

\[
a_0 = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) \, dx = \frac{1}{\ell} \int_{0}^{\ell} f(x) \, dx.
\]

Also, \( f(x) \cos \left( \frac{n\pi x}{\ell} \right) \) is an even function of \( x \) (the product of even functions is even), and therefore

\[
a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \left( \frac{n\pi x}{\ell} \right) \, dx = \frac{2}{\ell} \int_{0}^{\ell} f(x) \cos \left( \frac{n\pi x}{\ell} \right) \, dx, \quad n = 1, 2, \ldots.
\]

Finally, \( f(x) \sin \left( \frac{n\pi x}{\ell} \right) \) is an odd function of \( x \) (the product of an even and an odd function is odd), and thus

\[
b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \left( \frac{n\pi x}{\ell} \right) \, dx = 0, \quad n = 1, 2, \ldots.
\]

Thus the full Fourier series reduces to

\[
f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi x}{\ell} \right),
\]

where

\[
a_0 = \frac{1}{\ell} \int_{0}^{\ell} f(x) \, dx, \quad a_n = \frac{2}{\ell} \int_{0}^{\ell} f(x) \cos \left( \frac{n\pi x}{\ell} \right) \, dx, \quad n = 1, 2, \ldots.
\]

This is precisely the Fourier cosine series of \( f \) on the interval \((0, \ell)\) (cf. Section 6.2.2).
6.4 Finite element methods for the heat equation

1. We give the proof for the general case of a Gram matrix $G$. Suppose $G \mathbf{x} = \mathbf{0}$, where $\mathbf{x} \in \mathbb{R}^n$. Then $(\mathbf{x}, G \mathbf{x}) = 0$ must hold, and

\[
(x, Gx) = \sum_{i=1}^{n} (Gx)_i x_i = \sum_{i=1}^{n} \sum_{j=1}^{n} G_{ij} x_j x_i = \sum_{i=1}^{n} \sum_{j=1}^{n} (u_j, u_i)x_j x_i = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} x_j u_j, u_i \right) u_i
\]

\[
= \left( \sum_{j=1}^{n} x_j u_j, \sum_{i=1}^{n} x_i u_i \right) = \left\| \sum_{j=1}^{n} x_j u_j \right\|^2.
\]

Thus $G \mathbf{x} = \mathbf{0}$ implies that the vector $\sum_{j=1}^{n} x_j u_j$ is the zero vector. But, since $\{u_1, u_2, \ldots, u_n\}$ is linearly independent, this in turn implies that $x_1 = x_2 = \cdots = x_n = 0$, that is, that $\mathbf{x} = \mathbf{0}$. Since the only vector $\mathbf{x} \in \mathbb{R}^n$ satisfying $G \mathbf{x} = \mathbf{0}$ is the zero vector, $G$ is nonsingular.

3. (a) \[
M = \begin{bmatrix} 2/9 & 1/18 & 2/9 \\ 1/18 & 6 & -3 \\ 2/9 & -3 & 6 \end{bmatrix}, \quad K = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}, \quad f(t) = \frac{1}{162} \begin{bmatrix} 11 \cos(t) \\ 11 \cos(t) \end{bmatrix}.
\]

(b) The system of ODEs is $M \frac{d^2 u}{dt^2} = -K \alpha + f(t)$, that is,

\[
\frac{2}{9} \frac{d \alpha_1}{dt} + \frac{1}{18} \frac{d \alpha_2}{dt} = -6 \alpha_1 + 3 \alpha_2 + \frac{11 \cos(t)}{162},
\]

\[
\frac{1}{18} \frac{d \alpha_1}{dt} + \frac{2}{9} \frac{d \alpha_2}{dt} = 3 \alpha_1 - 6 \alpha_2 + \frac{11 \cos(t)}{162}.
\]

5. (a) Write $\rho_1 = 8.97$, $\rho_2 = 7.88$, and similarly for $c(x)$ and $\kappa(x)$. Then the IBVP is

\[
\rho(x)c(x) \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left( \kappa(x) \frac{\partial u}{\partial x} \right) = 0, \quad 0 < x < 100, \quad t > 0,
\]

\[
u(x,0) = 5, \quad 0 < x < 100,
\]

\[
u(0,t) = 0, \quad t > 0,
\]

\[
u(100,t) = 0, \quad t > 0.
\]

(b) The mass matrix $M$ is tridiagonal and symmetric, and its nonzero entries are

\[
M_{i,i+1} = \begin{cases} \frac{\rho_1 c_1 h}{6}, & i = 1, 2, \ldots, \frac{n}{2} - 1, \\ \frac{\rho_2 c_2 h}{6}, & i = \frac{n}{2}, \frac{n}{2} + 1, \ldots, n - 1, \end{cases}
\]

and

\[
M_{ii} = \begin{cases} \frac{2\rho_1 c_1 h}{3}, & i = 1, 2, \ldots, \frac{n}{2} - 1, \\ \frac{(\rho_1 c_1 + \rho_2 c_2) h}{3}, & i = \frac{n}{2}, \\ \frac{2\rho_2 c_2 h}{3}, & i = \frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n - 1. \end{cases}
\]

The stiffness matrix $K$ is tridiagonal and symmetric, and its nonzero entries are

\[
K_{i,i+1} = \begin{cases} -\frac{\kappa_1 h}{18}, & i = 1, 2, \ldots, \frac{n}{2} - 1, \\ -\frac{\kappa_2 h}{18}, & i = \frac{n}{2}, \frac{n}{2} + 1, \ldots, n - 1, \end{cases}
\]

and

\[
K_{ii} = \begin{cases} \frac{2\kappa_1 h}{18}, & i = 1, 2, \ldots, \frac{n}{2} - 1, \\ \frac{\kappa_1 + \kappa_2 h}{18}, & i = \frac{n}{2}, \\ \frac{2\kappa_2 h}{18}, & i = \frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n - 1. \end{cases}
\]
CHAPTER 6. HEAT FLOW AND DIFFUSION

7. The solution is

\[ u(x, t) = \sum_{n=1}^{\infty} a_n(t) \sin \left( \frac{n\pi x}{100} \right), \]

where

\[ a_n(t) = \frac{400(2 + (-1)^n)}{\kappa^2 \pi^7 n^7} \left( 10^4 \rho c \left( e^{-\kappa^2 \pi^2 t^{\frac{1}{2}}} - 1 \right) + \kappa n^2 \pi^2 t \right). \]

The errors in Examples 6.8 and 6.9 are graphed in Figure 6.4.

![Figure 6.4](image1)

Figure 6.4: The errors in Examples 6.8 and 6.9 (see Exercise 6.4.7).

9. (a) The IBVP is

\[ \rho(x)c(x) \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left( \kappa(x) \frac{\partial u}{\partial x} \right) = 0, \quad 0 < x < \ell, \quad t > 0, \]

\[ u(x, 0) = 5, \quad 0 < x < \ell, \]

\[ u(0, t) = 0, \quad t > 0, \]

\[ u(\ell, t) = 0, \quad t > 0. \]

(b) The weak form is to find \( u \) satisfying

\[ \int_0^t \left\{ \rho(x)c(x) \frac{\partial u}{\partial t}(x, t)v(x) + \kappa(x) \frac{\partial u}{\partial x}(x, t) \frac{dv}{dx}(x) \right\} \, dx = 0, \quad t > 0, \quad \forall v \in V. \]

(c) The temperature distribution after 120 seconds is shown in Figure 6.5.

![Figure 6.5](image2)

Figure 6.5: The temperature distribution after 120 seconds in Exercise 6.4.9.
6.5 Finite elements and Neumann conditions

1. The BVP to be solved is

\[-k \frac{d^2 u}{dx^2} = f(x), \quad 0 < x < 100, \]
\[\frac{du}{dx}(0) = 0,\]
\[\frac{du}{dx}(100) = 0,\]

with \(k = 1.5\) and \(f(x) = 10^{-7} x(25 - x)(100 - x) + 1/240\). The steady-state temperature is not unique; the solution with \(u(100) = 0\) is shown in Figure 6.6.

![Figure 6.6](image)

Figure 6.6: The computed steady-state temperature distribution from Exercise 6.5.1.

3. (a) The total amount of heat energy being added to the bar is 0.51 AW, where \(A\) is the cross-sectional area (0.01 AW through the left end and 0.5 AW in the interior). Therefore, 0.51 AW must be removed through the right end; that is, heat energy must be removed at a rate of 0.51 W/cm\(^2\) through the right end.

(b) The BVP is

\[-\kappa \frac{d^2 u}{dx^2} = 0.005, \quad 0 < x < 100, \]
\[k \frac{du}{dx}(0) = -0.01,\]
\[k \frac{du}{dx}(100) = -0.51.\]

The steady-state temperature is not unique; the temperature with \(u(100) = 0\) is graphed in Figure 6.7.

![Figure 6.7](image)

Figure 6.7: The computed steady-state temperature distribution from Exercise 6.5.3.
5. (a) We have

\[ \mathbf{u} \cdot \tilde{\mathbf{K}} \mathbf{u} = \sum_{i=0}^{n} \sum_{j=0}^{n} \tilde{K}_{ij} u_j u_i = \sum_{i=0}^{n} \sum_{j=0}^{n} a(\phi_j, \phi_i) u_j u_i = \sum_{i=0}^{n} \left( \sum_{j=0}^{n} u_j \phi_j, \phi_i \right) u_i = a(v, v). \]

Therefore,

\[ \tilde{\mathbf{K}} \mathbf{u} = 0 \Rightarrow \mathbf{u} \cdot \tilde{\mathbf{K}} \mathbf{u} = 0 \Rightarrow a(v, v) = 0. \]

(b) Suppose \( v \in \tilde{V} \) and \( a(v, v) = 0 \). By definition of \( a(\cdot, \cdot) \), this is equivalent to

\[ \int_{0}^{\ell} \kappa(x) \left( \frac{dv}{dx}(x) \right)^2 dx = 0. \]

Since the integrand is nonnegative, this implies that the integrand is in fact zero, and, since \( \kappa(x) \) is positive, we conclude that \( \frac{dv}{dx}(x) \) is the zero function. Therefore, \( v \) is a constant function.

(c) Thus, if \( \tilde{\mathbf{K}} \mathbf{u} = 0 \), it follows that \( v(x) = \sum_{i=0}^{n} u_i \phi_i(x) \) is a constant function, where \( u_0, u_1, \ldots, u_n \) are the components of \( \mathbf{u} \). But then there is a constant \( C \) such that \( v(x_i) = C, i = 0, 1, 2, \ldots, n \). These nodal values of \( v \) are precisely the numbers \( u_0, u_1, \ldots, u_n \), so we see that \( \mathbf{u} = C \mathbf{u}_c \), which is what we wanted to prove.

7. (a) Define \( \hat{V} = \{ v \in C^2[0, \ell] : v(0) = 0 \} \). Then the weak form of the BVP is

\[ \text{find } u \in \hat{V} \text{ such that } a(u, v) = (f, v) \text{ for all } v \in \hat{V}. \quad (6.1) \]

(b) The fact that a solution of the strong form is also a solution of the weak form is proved by the usual argument: multiply the differential equation by an arbitrary test function \( v \in \hat{V} \), and then integrate by parts.

The proof that a solution of the weak form is also a solution of the strong form is similar to the argument given in Section 6.5.2. Assuming that \( u \) satisfies the weak form (6.1), an integration by parts and some simplification shows that

\[ k(\ell) \frac{du}{dx}(\ell) v(\ell) - \int_{0}^{\ell} \left\{ \frac{d}{dx} \left[ k(x) \frac{du}{dx}(x) \right] + f(x) \right\} v(x) dx = 0 \text{ for all } v \in \hat{V}. \]

Since \( V \subset \hat{V} \), this implies that the differential equation

\[ \frac{d}{dx} \left[ k(x) \frac{du}{dx}(x) \right] + f(x) = 0, \quad 0 < x < \ell \]

holds, and we then have

\[ k(\ell) \frac{du}{dx}(\ell) v(\ell) = 0 \text{ for all } v \in \hat{V}. \]

Choosing any \( v \in \hat{V} \) with \( v(\ell) \neq 0 \) shows that the Neumann condition holds at \( x = \ell \).

9. The temperature distribution after 300 seconds is shown in Figure 6.8.
11. Here is a sketch of the proof: If \( \hat{K}u = 0 \), then

\[
u \cdot \hat{K}u = 0 \Rightarrow a(v, v) = 0,
\]

where \( v = \sum_{i=0}^{n-1} u_i \phi_i \). But then, by the usual reasoning, \( v \) must be a constant function, and \( v(x_n) = 0 \) (since \( \phi_i(x_n) = 0 \) for \( i = 0, 1, 2, \ldots, n - 1 \)). Thus \( v \) is the zero function, which implies that the nodal values of \( v \) are all zero. Therefore \( u = 0 \), and so \( \hat{K} \) is nonsingular.
Chapter 7

Waves

7.1 The homogeneous wave equation without boundaries

1. The solution, by d’Alembert’s formula, is \( u(x,t) = (\psi(x-ct) + \psi(x+ct)) \). Figure 7.1 shows three snapshots of \( u \).

![Figure 7.1: Three snapshots of the solution of Exercise 7.1.1: \( t = 0 \) (dashed curve), \( t = 0.01 \) (solid curve), and \( t = 0.025 \) (dash-dotted curve).](image)

3. The two waves join and add constructively, and then separate again. See Figure 7.2.

![Figure 7.2: Constructive interference in Exercise 7.1.3. The “blip” in the center is the result of two waves combining temporarily. (The velocity in this example is \( c = 1 \).)](image)
5. Consider the IVP

\[ \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad -\infty < x < \infty, \quad t > 0, \]
\[ u(x, 0) = \psi(x), \quad -\infty < x < \infty, \]
\[ \frac{\partial u}{\partial t} = 0, \quad -\infty < x < \infty, \]

where the support of \( \psi \) is \([a, b]\). The solution is \( u(x, t) = (\psi(x - ct) + \psi(x + ct))/2 \). Now, since \( \psi(x - ct) \) is a right-moving wave, we see that \( \psi(x_1 - ct) = 0 \) if any of the following conditions hold:

- \( x_1 \leq a; \)
- \( a < x_1 < b \) and \( t \geq \frac{x_1 - a}{c} \) (since \( \psi(x_1 - ct) = 0 \) after the trailing edge of the wave passes \( x_1 \));
- \( x \geq b \) and \( (t \leq \frac{x_1 - b}{c} \) or \( t \geq \frac{b - x_1}{c} \) (since \( \psi(x_1 - ct) = 0 \) before the leading edge of the wave reaches \( x_1 \) and after the trailing edge of the wave passes \( x_1 \)).

Similarly, \( \psi(x + ct) \) is a left-moving wave, and therefore \( \psi(x_1 + ct) = 0 \) if any of the following is true:

- \( x_1 \geq b; \)
- \( a < x_1 < b \) and \( t \geq \frac{b - x_1}{c} \);
- \( x_1 \leq a \) and \( (t \leq \frac{x_1 - a}{c} \) or \( t \geq \frac{b - x_1}{c} \)).

Since \( u(x_1, t) \) is guaranteed to be zero if both \( \psi(x_1 - ct) \) and \( \psi(x_1 + ct) \) are zero, we see that \( u(x_1, t) = 0 \) if any of the following is true:

- \( x_1 \leq a \) and \( (t \leq \frac{x_1 - a}{c} \) or \( t \geq \frac{b - x_1}{c} \));
- \( a < x_1 < b \) and \( t \geq \frac{x_1 - a}{c} \) and \( t \geq \frac{b - x_1}{c} \);
- \( x_1 \geq b \) and \( (t \leq \frac{x_1 - a}{c} \) or \( t \geq \frac{x_1 - a}{c} \)).

7. Consider the IBVP

\[ \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < \ell, \quad t > 0, \]
\[ u(x, 0) = \psi(x), \quad 0 < x < \ell, \]
\[ \frac{\partial u}{\partial t} = 0, \quad 0 < x < \ell, \]
\[ u(0, t) = 0, \quad t > 0, \]
\[ u(\ell, t) = 0, \quad t > 0, \]

where the support of \( \psi \) is \([a, b]\) and \( 0 < a < b < \ell \). If we define

\[ u(x, t) = \frac{1}{2} (\psi(x - ct) + \psi(x + ct)), \]

then we know that \( u \) satisfies the same PDE and initial condition as does the solution of the above IBVP. Also, since the support of \( \psi \) is \([a, b]\), \( u(0, t) = 0 \) for all \( t \leq a/c \) (since \( u(0, t) \) cannot be nonzero until the leading edge of the left-moving wave reaches \( x = 0 \)), and similarly \( u(\ell, t) = 0 \) for all \( t \leq (\ell - b)/c \) (since \( u(\ell, t) \) cannot be nonzero until the leading edge of the right-moving wave reaches \( x = \ell \)). Therefore, if \( t < tf = \min\{a/b, (\ell - b)/c\} \), then \( u \) satisfies both boundary conditions, in addition to the initial conditions and the PDE, and hence is a solution of the above IBVP.

### 7.2 Fourier series methods for the wave equation

1. The solution is found by setting \( c_n = 0 \) in Example 7.4:

\[ u(x, t) = \sum_{n=1}^{\infty} b_n \cos \left( \frac{cn\pi t}{25} \right) \sin \left( \frac{n\pi x}{25} \right). \]

A graph analogous to Figure 7.6 from the text is given in Figure 7.3.
4. The solution is

\[ u(x, t) = \sum_{n=1}^{\infty} a_n(t) \sin \left( \frac{(2n-1)\pi x}{50} \right), \]

where

\[ a_n(t) = b_n - \frac{50^2 c_n}{c^2(2n-1)^2\pi^2} \cos \left( \frac{c(2n-1)\pi t}{50} \right) + \frac{50^2 c_n}{c^2(2n-1)^2\pi^2} \]

and

\[ b_n = \frac{8 (\sqrt{2} \sin (n\pi/2) - \sqrt{2} \cos (n\pi/2) + (-1)^n)}{5(2n-1)^2\pi^2}, \quad c_n = \frac{4000}{(2n-1)\pi}. \]

The fundamental frequency is now \( c/100 \) instead of \( c/50 \). Fifty snapshots are shown in Figure 7.4.

5. The solution is

\[ u(x, t) = \left( \frac{1}{20} + 50t^2 \right) + \sum_{n=1}^{\infty} b_n \cos \left( \frac{cn\pi t}{25} \right) \cos \left( \frac{n\pi x}{25} \right), \]

where

\[ b_n = \frac{2 (2 \cos (n\pi/2) - 1 - (-1)^n)}{5n^2\pi^2}, \quad n = 1, 2, 3, \ldots. \]

Fifty snapshots of the solution are shown in Figure 7.5. The solution gradually moves up; this is possible because both ends are free to move vertically.

7. The solution is

\[ u(x, t) = \sum_{n=1}^{\infty} a_n(t) \sin \left( \frac{(2n-1)\pi x}{2} \right), \]
where
\[
a_n(t) = \left( b_n - \frac{4c_n}{c^2(2n-1)^2\pi^2} \right) \cos \left( \frac{c(2n-1)\pi t}{2} \right) + \frac{4c_n}{c^2(2n-1)^2\pi^2},
\]
\[
b_n = \frac{2(-1)^{n+1}}{125(2n-1)^2\pi^2},
\]
\[
c_n = \frac{4g}{(2n-1)\pi}.
\]

Snapshots of the solution are graphed in Figure 7.6, which should be compared to Figure 7.9 from the text.

9. Consider a string whose (linear) density (in the unstretched state) is 0.25 g/cm and whose longitudinal stiffness is 6500 N. Suppose the unstretched (zero tension) length of the string is 40 cm. We wish to determine the length to which the string must be stretched in order that its fundamental frequency be \( f_0 = 261 \) Hz (middle C). We will write \( \ell_0 = 40 \), \( k = 6.5 \cdot 10^8 \) (the stiffness of the string in g cm/s^2), and \( T \) for the unknown tension. Note that the mass of the string is \( m = 10 \) (g). We know that \( f_0 = c/2\ell_0 \), where \( c \) is the wave speed and \( \ell = \ell_0 + \Delta \ell \). Also, \( c^2 = T/\rho \), where \( \rho = m/\ell \). Finally, \( \Delta \ell \) is related to \( T \) by Hooke’s law: \( k\Delta \ell/\ell_0 = T \). Putting all these equations together, we obtain

\[
f_0 \begin{align*}
&= \frac{c}{2(\ell_0 + \Delta \ell)} = \frac{1}{2(\ell_0 + \Delta \ell)} \sqrt{\frac{k\Delta \ell}{\ell_0}} \\
&= \frac{\sqrt{k\Delta \ell}}{2\sqrt{m\ell_0(\ell_0 + \Delta \ell)}} \Rightarrow \frac{k\Delta \ell}{4m\ell_0(\ell_0 + \Delta \ell)} = f_0^2.
\end{align*}
\]

Some straightforward algebra yields

\[
\Delta \ell = \frac{4m\ell_0^2f_0^2}{k - 4m\ell_0f_0^2} \approx 8.0586.
\]

Thus the string must be stretched to 48.0586 cm.
7.3 Finite element methods for the wave equation

1. The fundamental period is $2\ell/c = 0.2$. Using $h = 100/20 = 5$, $Dt = T/60$, and the RK4 method, we obtain the solution shown in Figure 7.7.

2. Using $h = 100/20 = 5$, $Dt = T/60$, and the RK4 method, we obtain the solution shown in Figure 7.7.

3. (a) Since the initial disturbance is 24 cm from the boundary and the wave speed is 400 cm/s, it will take $24/400 = 0.06$ s for the wave to reach the boundary.

   (b) The IBVP is

   \[
   \begin{align*}
   \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} &= 0, \quad 0 < x < 50, \quad t > 0, \\
   u(x, 0) &= 0, \quad 0 < x < 50, \\
   \frac{\partial u}{\partial t}(x, 0) &= \gamma(x), \quad 0 < x < 50, \\
   u(0, t) &= 0, \quad t > 0, \\
   u(50, t) &= 0, \quad t > 0,
   \end{align*}
   \]

   with $c = 400$ and $\gamma$ given in the statement of the exercise.

   (c) Using piecewise linear finite elements and the RK4 method (with $h = 50/80$ and $\Delta t = 6 \cdot 10^{-4}$), we computed the solution over the interval $0 \leq t \leq 0.06$. Four snapshots are shown in Figure 7.8, which shows that it does take 0.06 seconds for the wave to reach the boundary. (The reader should notice the spurious “wiggles” in the computed solution; these are due to the fact that the true solution is not smooth.)

4. It is not possible to obtain a reasonable numerical solution using finite elements. In Figure 7.9, we display the result obtained using $h = 2.5 \cdot 10^{-3}$ and $\Delta t = 3.75 \cdot 10^{-4}$. Three snapshots are shown.

7. (a) The weak form is

   \[
   \int_0^t \left\{ \rho(x) \frac{\partial^2 u}{\partial t^2}(x, t)v(x) + k(x) \frac{\partial u}{\partial x}(x, t) \frac{dv}{dx}(x) \right\} dx = \int_0^t f(x, t)v(x) dx
   \]
for \( t \geq t_0 \) and \( v \in \tilde{V} \), where \( \tilde{V} = \{ v \in C^2[0, \ell] : v(0) = 0 \} \).

(b) The system of ODEs has the same form as in the case of a homogeneous bar:

\[
\tilde{M} \frac{d^2 u}{dt^2} + \tilde{K}u = \tilde{f}(t),
\]

except that the entries in the mass matrix are now

\[
\tilde{M}_{ij} = \int_{0}^{\ell} \rho(x) \phi_i(x) \phi_j(x) \, dx, \quad i, j = 1, 2, \ldots, n,
\]

and the entries in the stiffness matrix are now

\[
\tilde{K}_{ij} = \int_{0}^{\ell} k(x) \frac{d\phi_i}{dx}(x) \frac{d\phi_j}{dx}(x) \, dx, \quad i, j = 1, 2, \ldots, n.
\]

The components of the vector \( \tilde{f} \) are unchanged (see Section 7.3.2).

### 7.4 Resonance

1. We represent the solution as

\[
u(x, t) = \sum_{n=1}^{\infty} a_n(t) \sin \left( \frac{n\pi x}{\ell} \right).
\]

Following the usual procedure, \( a_n \) must solve the IVP

\[
d^2 a_n \left/ \frac{dt^2} \right. + \left( \frac{cn\pi}{\ell} \right)^2 a_n = c_n(t), \quad a_n(0) = 0, \quad \frac{da_n}{dt}(0) = 0,
\]

where \( c_n(t), n = 1, 2, \ldots, \) are the Fourier sine coefficients of the right-hand-side function \( f(t) = \sin(2\pi\omega t) \):

\[
c_n(t) = \frac{2}{\ell} \int_{0}^{\ell} \sin(2\pi\omega t) \sin \left( \frac{n\pi x}{\ell} \right) \, dx = \frac{2}{n\pi} \left( 1 - (-1)^n \right) \sin(2\pi\omega t)
\]

\[
= \begin{cases} 
0, & \text{if } n \text{ is odd,} \\
\frac{4}{n\pi} \sin(2\pi\omega t), & \text{if } n \text{ is even.}
\end{cases}
\]

Therefore, \( a_n(t) = 0 \) if \( n \) is even, while, if \( n \) is odd,

\[
a_n(t) = \frac{\ell}{cn\pi} \int_{0}^{t} \sin \left( \frac{cn\pi}{\ell} (t-s) \right) c_n(s) \, ds.
\]

If \( n \) is odd and \( \omega \neq cn/(2\ell) \), we obtain

\[
a_n(t) = \frac{4\ell^2 (cn \sin(2\pi\omega t) - 2\ell \omega \sin \left( \frac{cn\pi}{\ell} \right))}{cn^2 \pi^2 (c^2 n^2 - 4\ell^2 \omega^2)}.
\]
7.4. RESONANCE

If \( n \) is odd and \( \omega = cn/(2\ell) \), we obtain
\[
a_n(t) = \frac{2\ell(\ell \sin \left(\frac{cn\pi t}{\ell}\right) - cn\pi t \cos \left(\frac{cn\pi t}{\ell}\right))}{c^2 n^2 \pi^3}.
\]

Therefore, if \( \omega \) does not equal \( cn/(2\ell) \) for any odd \( n \), then the Fourier coefficients of \( u \) are uniformly bounded with respect to \( t \) and go to zero like \( 1/n^3 \). Notice, though, that if \( \omega \) is very close to \( cn/(2\ell) \) for some odd \( n \), then the corresponding \( a_n(t) \) is larger (due to the factor of \( c^2 n^2 - 4\ell^2 \omega^2 \) in the denominator), and thus that frequency has greater weight in the Fourier series.

On the other hand, if \( \omega = cn/(2\ell) \) for some odd \( n \), then the corresponding Fourier coefficient \( a_n(t) \) grows without bound as \( t \to \infty \), and resonance is observed.

3. The experiment described in this problem is modeled by the IBVP
\[
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < 1, \quad t > 0,
\]
\[
u(x, 0) = 0, \quad 0 < x < 1,
\]
\[
\frac{\partial u}{\partial t}(x, 0) = 0, \quad 0 < x < 1,
\]
\[
k \frac{\partial u}{\partial x}(0, t) = B \sin (2\pi \omega t), \quad t > 0,
\]
\[
u(1, t) = 0, \quad t > 0.
\]

Here \( c^2 = k/\rho \approx 3535.53 \text{ m/s} \).

We transform this problem into one that we can analyze by the Fourier series method by shifting the data. The function \( v(x, t) = (B/k) \sin (2\pi \omega t)(x - 1) \) satisfies the given boundary conditions; if we define \( w = u - v \), then \( w \) satisfies the IBVP
\[
\frac{\partial^2 w}{\partial t^2} - c^2 \frac{\partial^2 w}{\partial x^2} = \frac{4\pi^2 \omega^2 B}{k} \sin (2\pi \omega t)(x - 1), \quad 0 < x < 1, \quad t > 0,
\]
\[
w(x, 0) = 0, \quad 0 < x < 1,
\]
\[
\frac{\partial w}{\partial t}(x, 0) = -\frac{2\pi \omega B}{k} (x - 1), \quad 0 < x < 1,
\]
\[
\frac{\partial w}{\partial x}(0, t) = 0, \quad t > 0,
\]
\[
w(1, t) = 0, \quad t > 0.
\]

We solve this IBVP by the usual method. We write
\[
u(x, t) = \sum_{n=1}^{\infty} a_n(t) \sin (n\pi x).
\]

The Fourier coefficients \( a_n \) are found by solving the IVP
\[
d^2a_n}{dt^2} + c^2 n^2 \pi^2 a_n = c_n(t), \quad a_n(0) = 0, \quad \frac{da_n}{dt}(0) = b_n,
\]
where
\[
c_n(t) = 2 \int_0^1 \frac{4\pi^2 \omega^2 B}{k} \sin (2\pi \omega t)(x - 1) \sin (n\pi x) \, dx = -\frac{8\pi \omega^2 B}{kn} \sin (2\pi \omega t)
\]
and
\[
b_n = -\frac{4\pi \omega B}{k} \int_0^1 (x - 1) \sin (n\pi x) \, dx = \frac{4\omega B}{kn}.
\]

It follows (see Section 4.2.3 from the text) that
\[
a_n(t) = \frac{b_n}{cn\pi} \sin (cn\pi t) + \frac{1}{cn\pi} \int_0^t \sin (cn\pi(t - s))c_n(s) \, ds = \frac{4\omega B}{kcn^2 \pi} \sin (cn\pi t) + \frac{1}{cn\pi} \int_0^t \sin (cn\pi(t - s))c_n(s) \, ds.
\]

If \( \omega \neq cn/2 \), then
\[
a_n(t) = \frac{4\omega B}{kcn^2 \pi} \sin (cn\pi t) - \frac{8\omega^2 B(cn \sin (2\pi \omega t) - 2\omega \sin (cn\pi t))}{ckn^2 \pi(c^2 n^2 - 4\omega^2)}.
\]
while if $\omega = cn/2$,

$$a_n(t) = \frac{4\omega B}{kcn^2\pi} \sin (cn\pi t) - \frac{4\omega^2 B}{ckn^2\pi} \left( \frac{\sin (cn\pi t)}{cn\pi} - t \cos (cn\pi t) \right).$$

(a) From the above analysis, we see that the smallest resonant frequency is $\omega_r = c/2$, the fundamental frequency of the wave equation modeling the problem.

(b) If $\omega = \omega_r$, then $a_1(t)$ is given by the second formula above, while $a_n(t)$, $n > 1$, is given by the first. We compute several snapshots of the displacement, using 20 terms in the Fourier series, on each of the time intervals $[0, 0.005]$, $[0.005, 0.01]$, $[0.01, 0.015]$, and $[0.015, 0.02]$. These are shown in Figure 7.10. The graphs show that the amplitude grows as $t$ increase, and also that the shape of the displacement is dominated by the first fundamental mode.

![Figure 7.10: Resonance in Exercise 7.3.3(a).](image)

(c) With $\omega = \omega_r/4$, we compute the displacements at the same times as in the previous part of the exercise. The results are shown in Figure 7.11. Here the absence of resonance is clearly seen, in that the amplitudes remain bounded as $t$ increases.

![Figure 7.11: Absence of resonance in Exercise 7.3.3(b).](image)
5. We solve the IBVP by the usual method, taking
\[ u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin(n\pi x), \]
where the Fourier coefficient \( a_n \) solves the IVP
\[ \frac{d^2a_n}{dt^2} + c^2 n^2 \pi^2 a_n = c_n(t), \quad a_n(0) = 0, \quad \frac{da_n}{dt}(0) = 0. \]
Here \( c_n \) is the Fourier coefficient of the right-hand-side function \( f(x,t) \):
\[ c_n(t) = 2 \int_0^1 f(x,t) \sin(n\pi x) \, dx = 2 \sin(2\pi \omega t) \int_{3/5}^{13/5} \sin(n\pi x) \, dx \]
\[ = 2 \sin(2\pi \omega t) \frac{1}{n\pi} \left( \cos\left(\frac{3n\pi}{5}\right) - \cos\left(\frac{13n\pi}{20}\right) \right). \]
It follows that
\[ a_n(t) = \frac{1}{cn\pi} \int_0^t \sin(cn\pi(t-s))c_n(s) \, ds \]
\[ = \frac{2}{cn^2\pi^2} \left( \cos\left(\frac{3n\pi}{5}\right) - \cos\left(\frac{13n\pi}{20}\right) \right) \frac{cn\sin(2\pi \omega t) - 2\omega \sin(cn\pi t)}{\pi(c^2n^2 - 4\omega^2)}. \]
Five snapshots of the solution, at \( t = 0.002, 0.004, 0.006, 0.008, 0.01 \), are shown in Figure 7.12.

![Figure 7.12: Snapshots of the solution of Exercise 7.3.5.](image)

5.5 Finite difference methods for the wave equation

1. Let \( \phi : \mathbb{R} \to \mathbb{R} \) be at least twice continuously differentiable. Then, by Taylor’s theorem,
\[ \phi(x + \Delta x) = \phi(x) + \frac{d\phi}{dx}(x)\Delta x + \frac{1}{2} \frac{d^2\phi}{dx^2}(c)\Delta x^2, \]
where \( c \) is some number between \( x \) and \( x + \Delta x \). Solving for \( d\phi/dx \) yields
\[ \frac{d\phi}{dx}(x) = \frac{\phi(x + \Delta x) - \phi(x)}{\Delta x} - \frac{1}{2} \frac{d^2\phi}{dx^2}(c)\Delta x \]
or
\[ \frac{d\phi}{dx}(x) = \frac{\phi(x + \Delta x) - \phi(x)}{\Delta x} = \frac{1}{2} \frac{d^2\phi}{dx^2}(c)\Delta x. \]
Since
\[ \frac{d^2\phi}{dx^2}(c) \to \frac{d^2\phi}{dx^2}(x) \]
as \( \Delta x \to 0 \), it follows that the error in the forward difference approximation to the first derivative is approximately proportional to \( \Delta x \).
3. Suppose \( \phi : \mathbb{R} \to \mathbb{R} \) is at least four times continuously differentiable. By Taylor’s theorem,

\[
\phi(x + \Delta x) = \phi(x) + \frac{d\phi}{dx}(x)\Delta x + \frac{d^2\phi}{dx^2}(x)\Delta x^2 + \frac{d^3\phi}{dx^3}(x)\Delta x^3 + \frac{d^4\phi}{dx^4}(x)\Delta x^4,
\]

\[
\phi(x - \Delta x) = \phi(x) - \frac{d\phi}{dx}(x)\Delta x + \frac{d^2\phi}{dx^2}(x)\Delta x^2 - \frac{d^3\phi}{dx^3}(x)\Delta x^3 + \frac{d^4\phi}{dx^4}(x)\Delta x^4,
\]

where \( c_1 \) is between \( x \) and \( x + \Delta x \) and \( c_2 \) is between \( x \) and \( x - \Delta x \). Adding these two equations yields

\[
\phi(x + \Delta x) + \phi(x - \Delta x) = 2\phi(x) + \frac{d^2\phi}{dx^2}(x)\Delta x^2 - \frac{d^4\phi}{dx^4}(c_1)\Delta x^4 + \frac{d^4\phi}{dx^4}(c_2)\Delta x^4.
\]

Solving this equation for the second derivative yields

\[
\frac{d^2\phi}{dx^2}(x) = \frac{\phi(x + \Delta x) - 2\phi(x) + \phi(x - \Delta x)}{\Delta x^2} - \frac{1}{24}\left(\frac{d^4\phi}{dx^4}(c_1) + \frac{d^4\phi}{dx^4}(c_2)\right)\Delta x^2.
\]

Since

\[
\frac{d^4\phi}{dx^4}(c_1) + \frac{d^4\phi}{dx^4}(c_2) \to 2\frac{d^4\phi}{dx^4}(x)
\]

as \( \Delta x \to 0 \), it follows that the error in the central difference approximation to the second derivative is approximately proportional to \( \Delta x^2 \).

5. With a Dirichlet condition at the left endpoint and a Neumann condition at the right, we must compute

\[
u^{(i)}_k = u(x_i, t_j), \quad i = 1, 2, \ldots, k, \quad j = 1, 2, \ldots, n
\]

(using the same notation as in the text). We apply the usual 2-2 scheme at each \( (x_i, t_j) \), \( i = 1, 2, \ldots, k - 1 \), resulting in equation (7.41) (page 301 in the text) for computing \( u^{(i+1)}_k, \quad i = 1, 2, \ldots, k - 1 \). To compute \( u^{(j+1)}_k \), we approximate the Neumann condition

\[
\frac{\partial u}{\partial x}(x_k, t_j) = 0
\]

by

\[
\frac{u^{(i+1)}_k - u^{(i-1)}_k}{\Delta x} = 0 \quad \Rightarrow \quad u^{(i+1)}_k = u^{(i-1)}_k.
\]

We then obtain equation (7.48) for \( u^{(j+1)}_k \). Finally, we use (7.44) to compute \( u^{(1)}_i, \quad i = 1, 2, \ldots, k - 1 \), and (7.41) with \( i = k \) and \( \psi(x_{i+1}) \) replaced by \( \psi(x_{i-1}) \) to compute \( u^{(1)}_k \).

To test the scheme, we try in on the given problem, using a time interval of \([0, 1]\) with initial values of \( k = 10 \) and \( n = 500 \) (this gives \( c\Delta t/Dt = 0.9 \)). Doubling \( k \) and \( n \) at each iteration, we obtain the following maximum errors:

<table>
<thead>
<tr>
<th>( \Delta x )</th>
<th>( \Delta t )</th>
<th>maximum error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.002</td>
<td>( 1.9 \cdot 10^{-2} )</td>
</tr>
<tr>
<td>0.05</td>
<td>0.001</td>
<td>( 4.8802 \cdot 10^{-3} )</td>
</tr>
<tr>
<td>0.025</td>
<td>0.0005</td>
<td>( 1.2250 \cdot 10^{-3} )</td>
</tr>
<tr>
<td>0.0125</td>
<td>0.00025</td>
<td>( 3.0792 \cdot 10^{-4} )</td>
</tr>
</tbody>
</table>

These errors display the expected second-order behavior.
Chapter 8

First-order PDEs and the Method of Characteristics

8.1 The simplest PDE and the method of characteristics

1. Consider the PDE
   \[ \frac{\partial u}{\partial y} = 0. \]
   (a) Intuitively, since the solution is constant in \( y \), the characteristics must be the vertical lines \( x = \text{const.} \) We can verify this formally as follows. Suppose we impose the initial condition \( u(f(s), g(s)) = h(s) \). Then the characteristics are determined by the equations
   \[ \frac{\partial x}{\partial \tau} = 0, \quad x(0, \tau) = f(s), \]
   \[ \frac{\partial y}{\partial \tau} = 1, \quad y(0, \tau) = g(s) \]
   which yield
   \[ x(s, t) = f(s), \quad y(s, t) = g(s) + t. \]
   For each fixed \( s \), these equations parametrize the vertical line \( x = f(s) \).

   (b) The IVP
   \[ \frac{\partial u}{\partial y} = 0, \quad u(0, y) = \phi(y) \]
   is not well-posed. The general solution of the PDE is \( u(x, y) = \psi(x) \). The initial condition \( u(0, y) = \phi(y) \) then yields \( \psi(0) = \phi(y) \), which is only possible if \( \phi \) is a constant: \( \phi(y) = c \). Moreover, in this case, there are infinitely many solutions, one for each \( \psi \) satisfying \( \psi(0) = c \). Therefore the given IVP either has no solution (if \( \phi \) is not a constant function) or infinitely many (if \( \phi \) is a constant function).

3. Consider the PDE
   \[ 4 \frac{\partial u}{\partial x} + 3 \frac{\partial u}{\partial t} = 0, \quad 0 < x < 1, \quad t > 0. \]
   Given the initial condition \( u(f(s), g(s)) = h(s) \), the characteristics are determined by
   \[ \frac{\partial x}{\partial \tau} = 4, \quad x(0, \tau) = f(s), \]
   \[ \frac{\partial t}{\partial \tau} = 3, \quad t(0, \tau) = g(s), \]
   which yield
   \[ x(s, \tau) = 4\tau + f(s), \quad t(s, \tau) = 3\tau + g(s). \]
(a) Now consider the initial/boundary conditions
\[ u(x,0) = u_0(x), \quad 0 < x < 1, \]
\[ u(0,t) = \phi(t), \quad t > 0. \]

Characteristics passing through \((s,0), 0 < s < 1\), have the form
\[ x = 4\tau + s, \quad t = 3\tau \iff \tau = \frac{t}{3}, \quad s = x - \frac{4}{3}t. \]

This characteristic can also be written as \( t = (3/4)(x-s) \). On such characteristics,
\[ u(x,t) = v(s,\tau) = u_0(s) = u_0\left(x - \frac{4}{3}t\right). \]

Characteristics through \((0,s), s > 0\), have the form
\[ x = 4\tau, \quad t = 3\tau + s \iff \tau = \frac{x}{4}, \quad s = t - \frac{3}{4}x. \]

This characteristic can also be written as \( t = s + (3/4)x \). On such characteristics,
\[ u(x,t) = v(s,\tau) = \phi(s) = \phi\left(t - \frac{3}{4}x\right). \]

The characteristic dividing the domain into two is \( t = (3/4)x \). Thus
\[ u(x,t) = \begin{cases} 
  u_0\left(x - \frac{4}{3}t\right), & t \leq \frac{3}{4}x, \\
  \phi\left(t - \frac{3}{4}x\right), & t > \frac{3}{4}x.
\end{cases} \]

(b) Now consider the initial/boundary conditions
\[ u(x,0) = u_0(x), \quad 0 < x < 1, \]
\[ u(1,t) = \psi(t), \quad t > 0. \]

This set of conditions does not define a well-posed problem, since every characteristic through a point \((x_0,0), 0 < x_0 < 1\), also passes through a point \((1,t_0), 0 < t_0 < 3/4\). These means that the solution is over-determined on part of the domain, and \(u_0, \psi\) cannot be specified independently of each other.

5. Consider the IBVP
\[ \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0, \quad 0 < x < 1, \quad t > 0, \]
\[ u(x,0) = u_0(x), \quad 0 < x < 1, \]
\[ u(0,t) = \phi(t), \quad t > 0. \]

The solution is
\[ u(x,t) = \begin{cases} 
  u_0(x-ct), & t < c^{-1}x, \\
  \phi(t-c^{-1}x), & t > c^{-1}x.
\end{cases} \]

We assume that \(u_0\) and \(\phi\) are both continuously differentiable.

(a) We wish to determine the conditions on \(u_0, \phi\) that guarantee that \(u\) is continuous. Consider any \((x_0, t_0)\) with \(t_0 = c^{-1}x_0\). We have
\[ u_0(x-ct) \to u_0(x_0-ct_0) = u_0(0) \text{ as } (x,t) \to (x_0,t_0), \]
\[ \phi(t-c^{-1}x) \to \phi(t_0-c^{-1}x_0) = \phi(0) \text{ as } (x,t) \to (x_0,t_0), \]
which shows that \(u\) is continuous if and only if \(u_0(0) = \phi(0)\).
8.2. FIRST-ORDER QUASILINEAR PDEs

(b) Next, we have
\[ \frac{\partial u}{\partial x}(x,t) = \begin{cases} \frac{du_0}{dx}(x-ct), & t < c^{-1}x, \\ -c^{-1} \frac{d\phi}{dt}(t - c^{-1}x), & t > c^{-1}x, \end{cases} \]

and
\[ \frac{du_0}{dx}(x-ct) \rightarrow \frac{du_0}{dx}(x_0 - ct_0) = \frac{du_0}{dx}(0) \text{ as } (x,t) \rightarrow (x_0,t_0), \]
\[ -c^{-1} \frac{d\phi}{dt}(t - c^{-1}x) \rightarrow -c^{-1} \frac{d\phi}{dt}(t_0 - c^{-1}x_0) = -c^{-1} \frac{d\phi}{dt}(0) \text{ as } (x,t) \rightarrow (x_0,t_0). \]

Therefore, \( \partial u/\partial x \) is continuous if and only if
\[ \frac{du_0}{dx}(0) = -c^{-1} \frac{d\phi}{dt}(0). \]

Similar reasoning shows that this condition also ensures that \( \partial u/\partial t \) is continuous.

7. Consider the IVP
\[ \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} = x + y, \ (x,y) \in \mathbb{R}^2, \]
\[ u(x,-x) = x^2, \ x \in \mathbb{R}. \]

We solve the problem in characteristics variables as follows:
\[ \frac{\partial x}{\partial t} = 1, \ x(s,0) = s, \]
\[ \frac{\partial y}{\partial t} = 1, \ y(s,0) = -s, \]
\[ \frac{\partial v}{\partial t} = x + y, \ v(s,0) = s^2. \]

The first two equations yield
\[ x = s + t, \ y = -s + t \Leftrightarrow s = \frac{x-y}{2}, \ t = \frac{x+y}{2}, \]
and therefore
\[ \frac{\partial v}{\partial t} = 2t, \ v(s,0) = s^2 \Rightarrow v(s,t) = s^2 + t^2. \]

Thus,
\[ u(x,y) = \left( \frac{x-y}{2} \right)^2 + \left( \frac{x+y}{2} \right)^2 = \frac{1}{2} (x^2 + y^2). \]

8.2 First-order quasilinear PDEs

1. Consider the following IVP:
\[ x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} - (x+y)u = 0, \]
\[ u(1,y) = \phi(y). \]

(a) The characteristics are defined by the equations
\[ \frac{\partial x}{\partial t} = x, \ x(s,0) = 1, \]
\[ \frac{\partial y}{\partial t} = y, \ y(s,0) = s, \]
which yield
\[ x = e^t, \ y = se^t \Leftrightarrow t = \ln x, \ s = \frac{y}{x}. \]

Notice that \( x \) must be positive. Also, since \( s \) represents \( y_0 \) on the initial curve \( x = 1 \), we see that the characteristics can be written as \( y = y_0x \). Thus the characteristic through \((1, y_0)\) is the line with slope \( y_0 \) passing through the origin. This also shows that there must be a discontinuity when \( x \) decreases to zero.
(b) In the characteristic variables, the solution \( v(s,t) \) is defined by
\[
\frac{\partial v}{\partial t} = (x + y)v = (1 + s)e^tv, \quad v(s,0) = \phi(s).
\]
The ODE is separable and therefore the IVP is easily solved:
\[ v(s,t) = \phi(s)e^{(1+s)(e^t-1)} . \]
Changing variables, we obtain
\[ u(x,y) = \phi\left(\frac{y}{x}\right) e^{x+y-1-y/x} . \]

(c) The formula for \( u \) shows that it is defined for all \( x > 0 \) (or for all \( x < 0 \), but the initial curve lies in the right half-plane). This is consistent with the analysis of the characteristics.

3. Consider the IVP
\[
\frac{\partial u}{\partial x} + u \frac{\partial u}{\partial y} = 0, \quad u(x,0) = \phi(x).
\]
The PDE is quasi-linear and can be solved by solving the characteristic equations:
\[
\frac{\partial x}{\partial s} = v, \quad x(s,0) = s,
\]
\[
\frac{\partial y}{\partial s} = v, \quad y(s,0) = 0,
\]
\[
\frac{\partial v}{\partial s} = 0, \quad v(s,0) = \phi(s).
\]
The equations for \( v \) yield \( v(s,t) = \phi(s) \); substituting this into the first two equations yields \( x(s,t) = s + \phi(s)t \), \( y = \phi(s)t \). From this we obtain \( s = x - y, \ t = y/\phi(x-y) \), and the solution is
\[ u(x,y) = v(s,t) = \phi(s) = \phi(x-y), \]
that is, \( u(x,y) = \phi(x-y) \).
If we test the initial curve using the Jacobian test, we obtain
\[
\begin{vmatrix}
\frac{\partial x}{\partial s}(s,0) & \frac{\partial x}{\partial t}(s,0) \\
\frac{\partial y}{\partial s}(s,0) & \frac{\partial y}{\partial t}(s,0)
\end{vmatrix} = \begin{vmatrix} 1 & \phi(s) \\ 0 & \phi(s) \end{vmatrix} = \phi(s),
\]
we see that the initial curve is noncharacteristic if and only if \( \phi(s) \neq 0 \).

5. Consider the IVP
\[
\frac{\partial u}{\partial x} + u \frac{\partial u}{\partial y} = 0, \quad x > 0, \\
\quad u(x,0) = x, \quad x > 0.
\]
(a) The IVP can be solved by solving the characteristic equations:
\[
\frac{\partial x}{\partial s} = 1, \quad x(s,0) = s,
\]
\[
\frac{\partial y}{\partial s} = v, \quad y(s,0) = 0,
\]
\[
\frac{\partial v}{\partial s} = 0, \quad v(s,0) = s.
\]
It is straightforward to obtain the solution:
\[ x(s,t) = s + t, \quad y(s,t) = st, \quad v(s,t) = s. \]
Solving \( x = s + t, \ y = st \) for \( s, \ t \) (and bearing in mind that \( s = x \) when \( t = 0 \)), we obtain
\[ s = \frac{x + \sqrt{x^2 - 4y}}{2}, \ t = \frac{x - \sqrt{x^2 - 4y}}{2}. \]
8.3 Burgers’s equation

1. Consider the inviscid Burgers’s equation

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0 \]

with initial condition

\[ u(x, 0) = \frac{1}{1 + x^2}. \]

We have seen that the characteristics are the lines

\[ x = x_0 + u_0(x_0)t \quad \iff \quad t = u_0(x_0)^{-1}(x - x_0). \]

(a) We wish to show that the solution of the IVP is well-defined for at each point \((x, t), x < 0, t > 0\). To show this, it suffices to prove that there is a unique characteristic passing through any such point. Since the slope (in the \(x, t\) plane) of every characteristic is positive, the characteristic through \((x_0, 0)\) passes through \((x, t)\) only if \(x_0 < x < 0\) and

\[ t = (1 + x_0^2)(x - x_0) \quad \iff \quad x_0^2 - xx_0^2 + x_0 + (t - x) = 0. \]

If we define \(\psi(x_0) = x_0^2 - xx_0^2 + x_0 + (t - x)\), then it is easy to verify that \(\psi(0) > 0, \psi(x_0) < 0\) for \(x_0\) sufficiently large and negative, and \(\psi'(x_0) > 0\) for all \(x_0 < 0\). It follows that \(\psi(x_0) = 0\) has a unique solution \(x_0 < 0\), and hence there is a unique characteristic passing through \((x, t)\).

(b) We now wish to describe the set of all \((x, t), t > 0\), for which the solution is uniquely defined. Referring to the analysis in the text, culminating in Figure 8.9, we see that \(u(x, t)\) is uniquely defined for all \((x, t)\) outside the caustic, which is defined by the parametric equations

\[ x = \frac{1 + 3x_0^2}{2x_0}, \quad t = \frac{(1 + x_0^2)^2}{2x_0}, \quad x_0 > 0. \]

The corner of the caustic can be found by setting the derivatives of \(x\) and \(t\) (with respect to \(x_0\)) to zero and solving, which yields

\[ x_0 = \frac{1}{\sqrt{3}}, \quad x = \sqrt{3}, \quad t = \frac{8}{3\sqrt{3}}. \]

Moreover, we can solve

\[ x = \frac{1 + 3x_0^2}{2x_0} \]

for \(x_0\) in terms of \(x\) to get

\[ x_0 = \frac{x \pm \sqrt{x^2 - 3}}{3}. \]

Substituting into the equation for \(t\) yields the two branches of the caustic:

\[ t_1 = \frac{3 \left(1 + \left(\frac{x - \sqrt{x^2 - 3}}{3}\right)^2\right)^2}{2(x - \sqrt{x^2 - 3})}, \quad t_2 = \frac{3 \left(1 + \left(\frac{x + \sqrt{x^2 - 3}}{3}\right)^2\right)^2}{2(x + \sqrt{x^2 - 3})}, \quad x \geq \sqrt{3}. \]
where \( t_2 > t_1 \) for all \( x \). Comparing to Figure 8.9, we see that \( u(x,t) \) is uniquely defined if \( t > 0 \) and

\[
x < \sqrt{3} \text{ or } (x \geq \sqrt{3} \text{ and } t < t_1 \text{ or } t > t_2).
\]

3. Consider the IVP

\[
\frac{\partial u}{\partial t} + (1 - u) \frac{\partial u}{\partial x} = 0, \quad -\infty < x < 0, \quad t > 0,
\]

\[
u(x,0) = \frac{1}{x}, \quad -\infty < x < 0.
\]

(a) The characteristic equations are

\[
\frac{\partial x}{\partial \tau} = 1 - v, \quad x(s,0) = s,
\]

\[
\frac{\partial t}{\partial \tau} = 1, \quad t(s,0) = 0,
\]

\[
\frac{\partial v}{\partial \tau} = 0, \quad v(s,0) = \frac{1}{s}.
\]

Solving yields

\[
x(s,t) = s + \frac{s - 1}{s} \tau, \quad t(s,t) = \tau, \quad v(s,t) = \frac{1}{s}.
\]

Setting \( s = x_0 \), we obtain

\[
x = x_0 + \frac{x_0 - 1}{x_0} t \Leftrightarrow t = \frac{x_0}{x_0 - 1} (x - x_0).
\]

These are the equations of the characteristics.

(b) We can solve

\[
x = s + \frac{s - 1}{s} t
\]

for \( s \) as follows:

\[
x = s + \frac{s - 1}{s} t \Rightarrow s^2 + (t - x)s - t = 0
\]

\[
\Rightarrow s = \frac{x - t \pm \sqrt{(t - x)^2 + 4t}}{2}
\]

\[
\Rightarrow s = \frac{x - t - \sqrt{(t - x)^2 + 4t}}{2}.
\]

where the last step uses the fact that \( s \) must be negative. We then obtain \( u(x,t) = v(s,\tau) = 1/s \), that is,

\[
u(x,t) = \frac{2}{x - t - \sqrt{(t - x)^2 + 4t}}.
\]

(c) Notice that the slope of the characteristic decreases as \( x_0 \) increases; therefore, the characteristics through \((x_0,0)\) and \((x_1,0)\) will intersect at some point \((x,t)\) with \( t < 0 \). Solving the equations

\[
t = \frac{x_0}{x_0 - 1} (x - x_0), \quad t = \frac{x_1}{x_1 - 1} (x - x_1)
\]

for \((x,t)\) yields

\[
(x,t) = (x_0 - x_0 x_1 + x_1, -x_0 x_1).
\]

Notice that \( t < 0 \), as expected.

5. Consider the IVP

\[
\frac{\partial u}{\partial t} + a(u) \frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty, \quad t > 0,
\]

\[
u(x,0) = \phi(x), \quad x > 0.
\]

The characteristics are defined by

\[
x(s,\tau) = s + a(\phi(s)) \tau, \quad t(s,\tau) = \tau, \quad v(s,\tau) = \phi(s).
\]
We can write the characteristic passing through \((x_0, 0)\) as
\[
x = x_0 + a(\phi(x_0))t \iff t = \frac{x - x_0}{a(\phi(x_0))}.
\]

Assuming that \(a(\phi(x))\) is a decreasing function of \(x\), we know that characteristic passing through \((x_0, 0)\) will intersect with the characteristic passing through \((x_1, 0)\) if \(x_1 > x_0\). We can compute the point of intersection as follows:
\[
x = x_0 + a(\phi(x_0))t, \quad x = x_1 + a(\phi(x_1))t
\]
\[
\Rightarrow x_0 + a(\phi(x_0))t = x_1 + a(\phi(x_1))t
\]
\[
\Rightarrow t = \frac{x_0 - x_1}{a(\phi(x_1)) - a(\phi(x_0))} = -\frac{1}{a(\phi(x_1)) - a(\phi(x_0))}.
\]

If we take the limit as \(x_1\) decreases to \(x_0\), we find the smallest value of \(t\) for the characteristic through \((x_0, 0)\) intersects another characteristic:
\[
t^* = -\left. \frac{1}{\frac{dx}{ds}(\phi(x))} \right|_{x=x_0} = -\frac{1}{\frac{da}{du}(\phi(x_0)) \frac{d\phi}{dx}(x_0)}.
\]

On the other hand, suppose we look for the smallest value of \(\tau\) for which the change of variables from \((s, \tau)\) to \((x, t)\) is no longer well-defined. This occurs when the following Jacobian determinant is first zero:
\[
\begin{vmatrix}
\frac{\partial x}{\partial s}(s, \tau) & \frac{\partial x}{\partial \tau}(s, \tau) \\
\frac{\partial t}{\partial s}(s, \tau) & \frac{\partial t}{\partial \tau}(s, \tau)
\end{vmatrix} = 1 + \frac{da}{du}(\phi(s)) \frac{d\phi}{dx}(s) \tau \quad a(\phi(s)) \quad 1
\]
\[
= 1 + \frac{da}{du}(\phi(s)) \frac{d\phi}{dx}(s) \tau.
\]

Setting this expression equal to zero, solving for \(\tau\), and replacing \(s\) by \(x_0\) and \(\tau\) by \(t\) yields the same result as before.
Chapter 9

Green’s Functions

9.1 Green’s functions for BVPs in ODEs: Special cases

1. \( u(x) = xe^{-x} \).

3. (a) This BVP models a bar whose top end (originally at \( x = 0 \)) is free and whose bottom end is fixed at \( x = \ell \). If we apply a unit force to the cross-section at \( x = s \), then the part of the bar originally between \( x = s \) and \( x = \ell \) will compress, and the part of the bar originally above \( x = s \) will just move rigidly with \( u(x) = u(s) \) for \( 0 \leq x < s \). The compression of the bottom part of the bar will satisfy Hooke’s law:

\[
k \cdot u(x) = 1 \Rightarrow u(x) = \frac{\ell - x}{k}
\]

(the uncompressed length of the part of the bar between \( x = s \) and \( x = \ell \) is \( \ell - x \)). Therefore we obtain

\[
u(x) = \begin{cases} \frac{\ell - s}{k}, & 0 < x < s, \\ \frac{\ell - x}{k}, & s < x < \ell. \end{cases}
\]

The Green’s function is

\[
G(x; s) = \begin{cases} \frac{\ell - s}{k}, & 0 < x < s, \\ \frac{\ell - x}{k}, & s < x < \ell. \end{cases}
\]

(b) The Green’s function is

\[
G(x; s) = \int_{\max\{x, s\}}^{\ell} \frac{dz}{k(z)}.
\]

(c) If \( k \) is constant, then

\[
G(x; s) = \frac{\ell - \max\{x, s\}}{k} = \begin{cases} \frac{\ell - s}{k}, & 0 < x < s, \\ \frac{\ell - x}{k}, & s < x < \ell. \end{cases}
\]

(d) \( u(x) = \ln \left( \frac{2}{\sqrt{2} - \frac{x^2}{4} + \frac{x}{2} - \ln \frac{1 + x}{2}} \right) \).

5. By direct integration, we obtain the following solution to (9.11):

\[
u(x) = p \int_{0}^{x} \frac{ds}{k(s)}.
\]

Since \( x = \min\{x, \ell\} \) for \( x \in [0, \ell] \), we can write

\[
u(x) = p \int_{0}^{\min\{x, s\}} \frac{ds}{k(s)}.
\]

that is,

\[
u(x) = g(x; \ell)p.
\]
7. Let the operators $K : C^2_0[0, \ell] \to C[0, \ell]$ and $M : C[0, \ell] \to C^2_0[0, \ell]$ be defined by

$$Ku = -\frac{d}{dx} \left( k(x) \frac{du}{dx} \right)$$

and

$$(Mf)(x) = \int_0^x G(x;y)f(y) dy,$$

where

$$G(x;y) = \int_0^{\min\{x,y\}} \frac{1}{k(z)} dz.$$

We wish to prove directly that $(MK)u = u$ for all $u \in C^2_0[0, \ell]$. Given the definition of $G$, we will write

$$(Mf)(x) = \int_0^x G(x;y)f(y) dy + \int_x^\ell G(x;y)f(y) dy$$

$$= \int_0^x \int_0^\ell \frac{1}{k(z)} f(y) dz dy + \int_x^\ell \int_0^y \frac{1}{k(z)} f(y) dz dy$$

$$= \int_0^x \int_0^\ell \frac{1}{k(z)} f(y) dy dz + \int_x^\ell \int_0^\ell \frac{1}{k(z)} f(y) dy dz,$$

where the last step follows from changing the order of integration. Therefore,

$$(MKu)(x) = \int_0^x \int_0^x \frac{1}{k(z)} \left( k(x) \frac{du}{dx} (y) \right) dy dz - \int_0^x \int_x^\ell \frac{1}{k(z)} \left( k(x) \frac{du}{dx} (y) \right) dy dz$$

$$= \int_0^x \frac{1}{k(z)} \left[ k(x) \frac{du}{dx} (x) - k(z) \frac{du}{dx} (z) \right] dz - \int_0^x \frac{1}{k(z)} \left[ k(\ell) \frac{du}{dx} (\ell) - k(x) \frac{du}{dx} (x) \right] dz$$

$$= \int_0^x \frac{du}{dx} (z) dz = u(x)$$

(notice how we used both boundary conditions in the course of this calculation). This is the desired result.

### 9.2 Green’s functions for BVPs in ODEs: the symmetric case

1. The Green’s function is

$$G(x;y) = \begin{cases} \sin (\theta y)/((\tan \theta) \cos (\theta x) - \sin (\theta x)), & 0 \leq y \leq x, \\ \sin (\theta x)/((\tan \theta) \cos (\theta y) - \sin (\theta y)), & x \leq y \leq 1. \end{cases}$$

3. The Green’s function is

$$G(x;y) = \begin{cases} -kv_1(y)v_2(x), & 0 \leq y \leq x, \\ -kv_1(x)v_2(y), & x \leq y \leq 1. \end{cases}$$

where $v_1(x) = e^{\theta x} - e^{-\theta x}$, $u_2(x) = e^{\theta x} - e^{(2-\theta) x}$, and

$$k = \frac{1}{2(e^{2\theta} - 1)\theta}.$$
Suppose \( G \) is symmetric in the sense that \( G(y, x) = G(x, y) \) for all \( x, y \in [a, b] \). We then have

\[
(Mf, g)_{L^2} = \int_a^b (Mf)(x)g(x) \, dx = \int_a^b \left( \int_a^b G(x, y)f(y) \, dy \right) g(x) \, dx = \int_a^b \int_a^b G(x, y)f(y)g(x) \, dy \, dx = \int_a^b \int_a^b G(x, y)f(y)g(x) \, dx \, dy \text{ (changing the order of integration)}
\]

Thus \( M \) is a symmetric operator with respect to the \( L^2(a, b) \) inner product.

7. Define \( H : \mathbb{R} \to \mathbb{R} \) by

\[
H(x) = \begin{cases} 
0, & x < 0, \\
1, & 0 < x
\end{cases}
\]

We wish to prove that

\[
\int_{-\infty}^{\infty} \delta(x) \phi(x) \, dx = -\int_{-\infty}^{\infty} H(x) \frac{d\phi}{dx}(x) \, dx
\]

holds for all \( \phi \in \mathcal{D}(\mathbb{R}) \). That is, we wish to prove that if \( \phi \in \mathcal{D}(\mathbb{R}) \), then

\[
\int_{-\infty}^{\infty} H(x) \frac{d\phi}{dx}(x) \, dx = -\phi(0).
\]

This is a direct calculation. Choose \( b > 0 \) sufficiently large that \( \phi(x) = 0 \) for all \( x \geq b \). Then

\[
\int_{-\infty}^{\infty} H(x) \frac{d\phi}{dx}(x) \, dx = \int_{0}^{\infty} \frac{d\phi}{dx}(x) \, dx = \int_{0}^{b} \frac{d\phi}{dx}(x) \, dx = \phi(b) - \phi(0) = -\phi(0),
\]

as desired.

9. Let \( G \) be the Green’s function for the BVP

\[
-\frac{d}{dx} \left( P(x) \frac{du}{dx} \right) + R(x)u = f(x), \quad a < x < b,
\]

\[
\alpha_1 u(a) + \alpha_2 \frac{du}{dx}(a) = 0,
\]

\[
\beta_1 u(b) + \beta_2 \frac{du}{dx}(b) = 0.
\]

Then

\[
G(x; y) = \begin{cases} 
-\frac{v_1(y)v_2(x)}{P(x)v_2(x) - P(y)v_1(x)}, & a \leq y \leq x, \\
-\frac{v_1(x)v_2(y)}{P(x)v_2(x) - P(y)v_1(x)}, & x \leq y \leq b,
\end{cases}
\]

where \( v_1 \) and \( v_2 \) are certain solutions of the homogeneous ODE

\[
-\frac{d}{dx} \left( P(x) \frac{dv}{dx} \right) + R(x)v = 0, \quad a < x < b.
\]

We wish to show that, for fixed \( y \in (a, b) \), \( u(x) = G(x; y) \) is the solution of the above BVP with the right-hand-side function \( f(x) \) replaced by \( \delta(x - y) \). To do this, we must interpret the BVP in its weak sense, which is

\[
-\frac{d}{dx} \left( P(x) \frac{dv}{dx} \right)(x) + R(x)v(x) = \int_{a}^{b} \left\{ P(x) \frac{dv}{dx}(x) \int_{a}^{b} \delta(x - y)v(y) \, dx \right\} \, dx = v(y).
\]
Here \( u \) must satisfy the given boundary conditions and the above variational equation must hold for all test functions \( v \) satisfying the same boundary conditions. Now, the Green’s function is continuous but its derivative \( du/dx \) has a jump discontinuity at \( x = y \). We will write

\[
\frac{du}{dx}(y-) = \lim_{x \to y^-} \frac{du}{dx}(x) = -\frac{v_1(y)\alpha_2(y)}{P(y)W(y)},
\]

\[
\frac{du}{dx}(y+) = \lim_{x \to y^+} \frac{du}{dx}(x) = -\frac{v_1(y)\alpha_2(y)}{P(y)W(y)}.
\]

We now substitute \( u \) into the left-hand side of the variational equation to obtain

\[
-P(x)\frac{du}{dx}(x)v(x)\bigg|_a^b + \int_a^b \left( P(x)\frac{du}{dx}(x)\frac{dv}{dx}(x) + R(x)u(x)v(x) \right) \, dx
\]

\[
= P(a)\frac{du}{dx}(a)v(a) - P(b)\frac{du}{dx}(b)v(b) + \int_a^y \left( P(x)\frac{du}{dx}(x)\frac{dv}{dx}(x) + R(x)u(x)v(x) \right) \, dx
\]

\[
+ \int_y^b \left( P(x)\frac{du}{dx}(x)\frac{dv}{dx}(x) + R(x)u(x)v(x) \right) \, dx.
\]

The integrands of both integrals on smooth on the intervals of integration, and hence integration by parts is valid:

\[
-P(x)\frac{du}{dx}(x)v(x)\bigg|_a^b + \int_a^b \left( P(x)\frac{du}{dx}(x)\frac{dv}{dx}(x) + R(x)u(x)v(x) \right) \, dx
\]

\[
= P(a)\frac{du}{dx}(a)v(a) - P(b)\frac{du}{dx}(b)v(b) + P(y)\frac{du}{dx}(y^-)v(y) - P(a)\frac{du}{dx}(a)v(a) + P(b)\frac{du}{dx}(b)v(b) - P(y)\frac{du}{dx}(y^+)v(y)
\]

\[
+ \int_a^y \left\{ -\frac{d}{dx} \left( P(x)\frac{du}{dx}(x) \right) v(x) + R(x)u(x)v(x) \right\} \, dx + \int_y^b \left\{ -\frac{d}{dx} \left( P(x)\frac{du}{dx}(x) \right) v(x) + R(x)u(x)v(x) \right\} \, dx
\]

\[
= P(y)\frac{du}{dx}(y^-)v(y) - P(y)\frac{du}{dx}(y^+)v(y) + \int_a^b \left\{ -\frac{d}{dx} \left( P(x)\frac{du}{dx}(x) \right) + R(x)u(x) \right\} v(x) \, dx
\]

\[
= P(y)\frac{du}{dx}(y^-)v(y) - P(y)\frac{du}{dx}(y^+)v(y),
\]

where the last step follows because, on both \((a, y)\) and \((y, b)\), \( u(x) = G(x; y) \) is a smooth solution of the homogeneous ODE

\[
-\frac{d}{dx} \left( P(x)\frac{du}{dx} \right) + R(x)u = 0.
\]

We now have

\[
-P(x)\frac{du}{dx}(x)v(x)\bigg|_a^b + \int_a^b \left( P(x)\frac{du}{dx}(x)\frac{dv}{dx}(x) + R(x)u(x)v(x) \right) \, dx
\]

\[
= P(y)v(y)\left( \frac{du}{dx}(y^-) - \frac{du}{dx}(y^+) \right)
\]

\[
= P(y)v(y)\left( \alpha_1 u(a) + \alpha_2 \frac{du}{dx}(a) = v_a, \right)
\]

\[
\beta_1 u(b) + \beta_2 \frac{du}{dx}(b) = v_b,
\]

11. We wish to find a formula for the solution of

\[
-\frac{d}{dx} \left( P(x)\frac{du}{dx} \right) + R(x)u = 0, \ a < x < b,
\]

\[
\alpha_1 u(a) + \alpha_2 \frac{du}{dx}(a) = v_a,
\]

\[
\beta_1 u(b) + \beta_2 \frac{du}{dx}(b) = v_b,
\]
in terms of the Green’s function \( G \) for the BVP
\[
-\frac{d}{dx} \left( P(x) \frac{du}{dx} \right) + R(x)u = f(x), \quad a < x < b,
\]
\[
\alpha_1 u(a) + \alpha_2 \frac{du}{dx}(a) = 0,
\]
\[
\beta_1 u(b) + \beta_2 \frac{du}{dx}(b) = 0.
\]

We proceed as in Section 9.2.2; in particular, we apply
\[
(u, L v) - (L u, v) = P(a) \left( u(a) \frac{dv}{dx}(a) - \frac{du}{dx}(a)v(a) \right) - P(b) \left( u(b) \frac{dv}{dx}(b) - \frac{du}{dx}(b)v(b) \right)
\]
as in the text (where \( L \) is defined by (9.16), with \( v(x) = G(x; y) \) (for a fixed \( y \)) and \( u \) the solution of the above BVP with inhomogeneous boundary conditions. We then have \((L v)(x) = \delta(x - y)\), and hence \((u, L v) = u(y)\).

Also, \( Lu = 0 \) since \( u \) satisfies the homogeneous ODE, and therefore \((L u, v) = 0\). We thus obtain
\[
u(y) = P(a) \left( u(a) \frac{dv}{dx}(a) - \frac{du}{dx}(a)v(a) \right) - P(b) \left( u(b) \frac{dv}{dx}(b) - \frac{du}{dx}(b)v(b) \right).
\]

We now apply the boundary condition satisfied by \( u \) and \( v \):
\[
\alpha_1 u(a) + \alpha_2 \frac{du}{dx}(a) = v_a \Rightarrow \frac{du}{dx}(a) = \frac{1}{\alpha_2} v_a - \frac{\alpha_1}{\alpha_2} u(a),
\]
\[
\beta_1 u(b) + \beta_2 \frac{du}{dx}(b) = v_b \Rightarrow \frac{du}{dx}(b) = \frac{1}{\beta_2} v_b - \frac{\beta_1}{\beta_2} u(b),
\]
\[
\alpha_1 v(a) + \alpha_2 \frac{dv}{dx}(a) = 0 \Rightarrow \frac{dv}{dx}(a) = -\frac{\alpha_1}{\alpha_2} v(a),
\]
\[
\beta_1 v(b) + \beta_2 \frac{dv}{dx}(b) = 0 \Rightarrow \frac{dv}{dx}(b) = -\frac{\beta_1}{\beta_2} v(b).
\]

We thus obtain
\[
u(y) = P(a) \left( u(a) \frac{dv}{dx}(a) - \frac{du}{dx}(a)v(a) \right) - P(b) \left( u(b) \frac{dv}{dx}(b) - \frac{du}{dx}(b)v(b) \right)
\]
\[
= P(a) \left( -\frac{\alpha_1}{\alpha_2} u(a)v(a) - \frac{1}{\alpha_2} v_a v(a) + \frac{\alpha_1}{\alpha_2} u(a)v(a) \right) - P(b) \left( -\frac{\beta_1}{\beta_2} u(b)v(b) - \frac{1}{\beta_2} v_b v(b) + \frac{\beta_1}{\beta_2} u(b)v(b) \right)
\]
\[
= \beta_2^{-1} P(b)v(b)v_b - \alpha_2^{-1} P(a)v(a)v_a
\]
\[
= \beta_2^{-1} P(b)G(b; y)v_b - \alpha_2^{-1} P(a)G(a; y)v_a
\]
\[
= \beta_2^{-1} P(b)G(y; b)v_b - \alpha_2^{-1} P(a)G(y; a)v_a.
\]

In the last step we used the symmetry of \( G \). This is the desired formula:
\[
u(y) = \beta_2^{-1} P(b)G(y; b)v_b - \alpha_2^{-1} P(a)G(y; a)v_a.
\]

### 9.3 Green’s functions for BVPs in ODEs: the general case

1. The Green’s function is
\[
G(x; y) = \begin{cases} 
-\frac{v_1(y)u_1(x)}{P(x)W(x)}, & 0 \leq y \leq x, \\
-\frac{v_1(y)v_2(x)}{P(x)W(x)}, & x \leq y \leq 1,
\end{cases}
\]
where \( v_1(x) = e^x - e^{2x}, \ v_2(x) = e^x - e^{2x}^{-1} \), and \( P(x)W(x) \) is the constant \( 1 - e^{-1} \). The solution to the BVP is
\[
u(x) = \int_0^1 G(x; y)f(y)w(y) \, dy,
\]
where \( w(x) = e^{-3x} \).
3. The Green’s function is
\[ G(x; y) = \begin{cases} \frac{v_1(x)w_2(x)}{P(x)W(x)}, & 0 \leq y \leq x, \\ \frac{v_2(x)w_1(x)}{P(x)W(x)}, & x \leq y \leq 1, \end{cases} \]
where \( v_1(x) = e^x \sin(2x), \) \( v_2(x) = e^x (\sin(2x) - (\tan(2)) \cos(2x)), \) and \( P(x)W(x) \) is the constant \( 2 \tan(2). \) The solution to the BVP is
\[ u(x) = \int_0^1 G(x; y)f(y)w(y) \, dy, \]
where \( w(x) = e^{-2x}. \)

5. The Green’s function is
\[ G(x; y) = \begin{cases} e^y (6 + 6x + 3x^2 + x^3), & \frac{1}{2} \leq y \leq x, \\ e^x (6 + 6y + 3y^2 + y^3), & x \leq y \leq 1. \end{cases} \]
The solution to the BVP when \( f(x) = x \) is
\[ u(x) = \frac{23}{2} + \frac{23}{2} x + 6x^2 + 2x^3 - 7e^{x-1}. \]

7. Let \( G \) be the Green’s function derived in this section, and let \( M \) be defined by \( (M f)(x) = \int_a^b G(x; y)f(y) \, dy. \) (Here \( M \) maps complex \( C[a, b] \) into complex \( C^2[a, b]. \) ) We wish to prove that \( (M f, g)_w = (f, Mg)_w \) for all \( f, g \in C[a, b]. \) In the following calculation, we use the facts that \( G \) is symmetric \((G(x; y) = G(y; x) \text{ for all } x, y \in [a, b])\) and \( G \) is real-valued, so that \( G(x; y) = G(x; y) \). We have
\[
(M f, g)_w = \int_a^b (M f)(x)\overline{g(x)} w(x) \, dx = \int_a^b \left( \int_a^b G(x; y)f(y)\overline{g(y)} w(y) \, dy \right) \overline{g(x)} w(x) \, dx \\
= \int_a^b \int_a^b G(x; y)f(y)\overline{g(y)}w(y) \, dy \, dx \\
= \int_a^b \int_a^b G(x; y)f(y)\overline{g(y)}w(x) \, dx \, dy \\
= \int_a^b f(y) \left( \int_a^b G(x; y)\overline{g(x)}w(x) \, dx \right) \overline{g(y)} w(y) \, dy \\
= \int_a^b f(y) \left( \int_a^b G(y; x)g(x)w(x) \, dx \right) \overline{g(y)} w(y) \, dy \\
= \int_a^b f(y) \left( \int_a^b G(y; x)g(x)w(x) \, dx \right) \overline{g(y)} w(y) \, dy \\
= \int_a^b f(y)\overline{(M g)(y)} w(y) \, dy = (f, Mg)_w.
\]
(Notice how the key step was the interchanging of the order of integration.) This proves the desired result.

**9.4 Introduction to Green’s functions for IVPs**

1. We wish to show that the solution of
\[ \frac{d^2u}{dt^2} + \theta^2 u = f(t), \quad u(0) = u_0, \quad \frac{du}{dt}(0) = v_0 \]
is
\[ u(t) = \frac{\partial G}{\partial t}(t; 0)u_0 + G(t; 0)v_0 + \int_0^t G(t; s)f(s) \, ds, \]
where
\[ G(t; s) = \begin{cases} \frac{\sin(\theta(t-s))}{\theta}, & t > s, \\ 0, & t < s. \end{cases} \]
We can write the proposed solution explicitly as
\[ u(t) = \cos(\theta t)u_0 + \frac{\sin(\theta t)}{\theta}v_0 + \int_0^t \frac{\sin(\theta(t-s))}{\theta} f(s) \, ds. \]
9.4. INTRODUCTION TO GREEN'S FUNCTIONS FOR IVPS

Then

\[
\frac{du}{dt} (t) = -\theta \sin (\theta t) u_0 + \cos (\theta t) v_0 + \frac{\sin (\theta (t - t))}{\theta} f(t) + \int_0^t \cos (\theta (t - s)) f(s) \, ds
\]

\[
= -\theta \sin (\theta t) u_0 + \cos (\theta t) v_0 + \int_0^t \cos (\theta (t - s)) f(s) \, ds
\]

\[
\frac{d^2 u}{dt^2} (t) = -\theta^2 \cos (\theta t) u_0 - \theta \sin (\theta t) v_0 + \cos (\theta (t - t)) f(t) - \int_0^t \theta \sin (\theta (t - s)) f(s) \, ds
\]

\[
= -\theta^2 \cos (\theta t) u_0 - \theta \sin (\theta t) v_0 + f(t) - \int_0^t \theta \sin (\theta (t - s)) f(s) \, ds.
\]

From these formulas, we can easily verify that \( u \) satisfies the ODE and the initial conditions:

\[
u(0) = \cos (0) u_0 + \frac{\sin (0)}{\theta} v_0 + \int_0^0 \sin (\theta (t - s)) f(s) \, ds = u_0,\]

\[
\frac{du}{dt} (0) = -\theta \sin (0) u_0 + \cos (0) v_0 + \int_0^0 \cos (\theta (t - s)) f(s) \, ds = v_0,
\]

and

\[
\frac{d^2 u}{dt^2} (t) + \theta^2 u(t) = -\theta^2 \cos (\theta t) u_0 - \theta \sin (\theta t) v_0 + f(t) - \int_0^t \theta \sin (\theta (t - s)) f(s) \, ds + \theta^2 \cos (\theta t) u_0 + \theta \sin (\theta t) v_0 + \int_0^t \theta \sin (\theta (t - s)) f(s) \, ds
\]

\[
= f(t).
\]

3. The Green’s function is

\[
G(t; s) = \begin{cases} \frac{\sin (2(t-s))}{2}, & t > s, \\ 0, & t < s. \end{cases}
\]

5. The Green’s function is

\[
G(t; s) = \begin{cases} e^{-(t-s)} - e^{-2(t-s)}, & t > s, \\ 0, & t < s. \end{cases}
\]

7. The Green’s function is

\[
G(t; s) = \begin{cases} e^{0.02(t-s)}, & t > s, \\ 0, & t < s, \end{cases}
\]

and the solution to the IVP is

\[
P(t) = G(t; 0) P_0 + \int_0^\infty G(t; s) f(s) \, ds = 55.5 e^{0.02t} + 450 + 10t - 450 e^{0.02t} = 450 + 10t - 394.5 e^{0.02t}.
\]

9. Let the solution of

\[
\frac{d^2 u}{dt^2} + a_1(t) \frac{du}{dt} + a_0(t) u = 0,
\]

\[
u(s) = 0,
\]

\[
\frac{du}{dt} (s) = f(s)
\]

be \( u(t) = S(t; s) f(s) \), and define the Green’s function \( G(t; s) \) by

\[
G(t; s) = \begin{cases} S(t; s), & t > s, \\ 0, & t < s, \end{cases}
\]

We wish to show that \( G \) satisfies

\[
\frac{\partial^2 G}{\partial t^2} (t; s) + a_1(t) \frac{\partial G}{\partial t} (t; s) + a_0(t) G(t; s) = \delta(t - s),
\]

\[
G(t_0; s) = 0,
\]

\[
\frac{\partial G}{\partial t} (t_0; s) = 0,
\]
provided \( t_0 < s \). We notice that \( G(t, s) = H(t - s)S(t, s) \), where \( H \) is the Heaviside function. We will show that
\[
\frac{\partial G}{\partial t}(t, s) = H(t - s) \frac{\partial S}{\partial t}(t, s) \quad \text{and} \quad \frac{\partial^2 G}{\partial t^2}(t, s) = \delta(t - s) + H(t - s) \frac{\partial^2 S}{\partial t^2}(t, s)
\]
(in the weak sense). Let \( \phi \) be any smooth function having compact support in \([t_0, \infty)\), that is, any function for which \( \phi(t) = 0 \) for all \( t \notin (a, b) \), where \( t_0 < a < b < \infty \). We first show that
\[
- \int_{t_0}^{\infty} G(t, s) \frac{d\phi}{dt}(t) dt = \int_{t_0}^{\infty} H(t - s) \frac{\partial S}{\partial t}(t, s) \phi(t) dt,
\]
which implies that
\[
\frac{\partial G}{\partial t}(t, s) = H(t - s) \frac{\partial S}{\partial t}(t, s)
\]
in the weak sense. We have
\[
- \int_{t_0}^{\infty} G(t, s) \frac{d\phi}{dt}(t) dt = - \int_{t_0}^{\infty} H(t - s)S(t, s) \frac{d\phi}{dt}(t) dt
\]
\[
= - \int_{s}^{b} S(t, s) \frac{d\phi}{dt}(t) dt
\]
\[
= - S(t, s) \phi(t)|_{s}^{b} \quad \text{by integration by parts}
\]
\[
= \int_{s}^{b} \frac{\partial S}{\partial t}(t, s) \phi(t) dt \quad \text{(since } S(s, s) = 0 \text{ and } \phi(b) = 0) \]
\[
= \int_{t_0}^{\infty} H(t - s) \frac{\partial S}{\partial t}(t, s) \phi(t) dt.
\]
This proves the desired formula for \( \partial G/\partial t \).

Next, we will show that
\[
- \int_{t_0}^{\infty} \frac{\partial G}{\partial t}(t, s) \frac{d\phi}{dt}(t) dt = \phi(s) + \int_{t_0}^{\infty} H(t - s) \frac{\partial^2 S}{\partial t^2}(t, s) \phi(t) dt,
\]
which implies that
\[
\frac{\partial^2 G}{\partial t^2}(t, s) = \delta(t - s) + H(t - s) \frac{\partial^2 S}{\partial t^2}(t, s)
\]
in the weak sense. We have
\[
- \int_{t_0}^{\infty} \frac{\partial G}{\partial t}(t, s) \frac{d\phi}{dt}(t) dt = - \int_{t_0}^{\infty} H(t - s) \frac{\partial S}{\partial t}(t, s) \frac{d\phi}{dt}(t) dt
\]
\[
= - \int_{s}^{b} \frac{\partial S}{\partial t}(t, s) \frac{d\phi}{dt}(t) dt
\]
\[
= - \frac{\partial S}{\partial t}(t, s) \phi(t)|_{s}^{b} \quad \text{by integration by parts}
\]
\[
= \phi(s) + \int_{s}^{b} \frac{\partial^2 S}{\partial t^2}(t, s) \phi(t) dt \quad \text{(since } \frac{\partial S}{\partial t}(s, s) = 1 \text{ and } \phi(b) = 0) \]
\[
= \phi(s) + \int_{t_0}^{\infty} H(t - s) \frac{\partial^2 S}{\partial t^2}(t, s) \phi(t) dt.
\]
This proves the desired formula for \( \partial G^2/\partial t^2 \).

Since \( S(t, s) \), as a function of \( t \), satisfies the homogeneous version of the ODE, we see that
\[
\frac{\partial^2 G}{\partial t^2}(t, s) + a_1(t) \frac{\partial G}{\partial t}(t, s) + a_0(t)G(t; s) = \delta(t - s),
\]
as desired. Also,
\[
G(t_0, s) = H(t_0 - s)S(t_0, s) = 0, \quad \frac{\partial G}{\partial t}(t_0, s) = H(t_0 - s) \frac{\partial S}{\partial t}(t_0, s) = 0
\]
since \( H(t_0 - s) = 0 \) because \( t_0 - s < 0 \) by assumption. This completes the proof.
9.5 Green’s functions for the heat equation

1. Given that

\[ S(x,t) = \frac{1}{2\sqrt{\pi kt}} e^{-x^2/(4kt)}, \]

we have

\[ \int_{-\infty}^{\infty} S(x,t) \, dx = \frac{1}{2\sqrt{\pi kt}} \int_{-\infty}^{\infty} e^{-u^2} \, du = \frac{1}{\sqrt{\pi}} \sqrt{\pi} = 1. \]

Making the change of variables \( u = x/(2\sqrt{kt}) \), we obtain

\[ \int_{-\infty}^{\infty} S(x,t) \, dx = \int_{-\infty}^{\infty} e^{-u^2} \, du = \frac{1}{\sqrt{\pi}} \sqrt{\pi} = 1. \]

Also, for any \( x,y \in \mathbb{R} \),

\[ \int_{-\infty}^{\infty} S(x-y,t) \, dx = \int_{-\infty}^{\infty} S(v,t) \, dv = 1. \]

3. For \( t = 660 \), six terms of the Fourier series are sufficient to give an accurate graph. For \( t = 61 \), even 100 terms are not sufficient to give a correct graph.

5. Following the example in Section 9.5.2, we obtain the following Green’s function:

\[ G(x,t; 75, 60) = \begin{cases} \frac{2}{\ell \rho c} \sum_{n=1}^{\infty} e^{-kn^2 \pi^2 t/\ell^2} \sin \left( \frac{(2n-1)\pi x}{2\ell} \right) \sin \left( \frac{(2n-1)\pi y}{2\ell} \right), & 0 \leq s \leq t, \\ \frac{2}{\ell \rho c} \sum_{n=1}^{\infty} \frac{1}{\ell} \int_{0}^{\ell} \sin \left( \frac{n\pi y}{\ell} \right) \phi(y) \, dy \, e^{-kn^2 \pi^2 t/\ell^2} \sin \left( \frac{n\pi x}{\ell} \right), & s > t. \end{cases} \]

Here we used the correct eigenfunctions for the given mixed boundary conditions and also adjust for the fact that the constants appear differently in the PDE in this exercise than in the example in Section 9.5.2 (notice the extra factor of \( 1/(\rho c) \) in the formula for this Green’s function). As before, \( k = \kappa/(\rho c) \). Snapshots of \( G(x,t; 75, 60) \) are shown in Figure 9.1.

![Graph of G(x,t;75,60) for various values of t](image)

Figure 9.1: Snapshots of the Green’s function in Exercise 9.5.5 at times \( t = 120, 240, 360, 480, 600 \) seconds. Twenty terms of the Fourier series were used to create these graphs.

7. We have

\[ u(x,t) = \int_{0}^{t} G(x,t; y, 0) \phi(y) \, dy \]

\[ = \int_{0}^{t} \frac{2}{\ell} \sum_{n=1}^{\infty} e^{-kn^2 \pi^2 t/\ell^2} \sin \left( \frac{n\pi y}{\ell} \right) \sin \left( \frac{n\pi x}{\ell} \right) \phi(y) \, dy \]

\[ = \sum_{n=1}^{\infty} \left( \frac{2}{\ell} \int_{0}^{\ell} \sin \left( \frac{n\pi y}{\ell} \right) \phi(y) \, dy \right) e^{-kn^2 \pi^2 t/\ell^2} \sin \left( \frac{n\pi x}{\ell} \right) \]

\[ = \sum_{n=1}^{\infty} b_n e^{-kn^2 \pi^2 t/\ell^2} \sin \left( \frac{n\pi x}{\ell} \right), \]
CHAPTER 9. GREEN’S FUNCTIONS

9.6 Green’s functions for the wave equation

1. Let

\[ u(x, t) = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y, s) \, dy \, ds. \]

We wish to show that \( u \) satisfies the inhomogeneous wave equation (with right-hand side \( f(x, t) \)) on the real line, subject to zero boundary conditions. We have

\[
\frac{\partial u}{\partial t}(x, t) = \frac{1}{2c} \int_{x-c(t-t)}^{x+c(t-t)} f(y, t) \, dy + \frac{1}{2c} \int_0^t \{ cf(x + c(t-s), s) + cf(x - c(t-s), s) \} \, ds
\]

\[
= \frac{1}{2c} \int_x^x f(y, t) \, dy + \frac{1}{2c} \int_0^t \{ cf(x + c(t-s), s) + cf(x - c(t-s), s) \} \, ds
\]

\[
= \frac{1}{2c} \int_0^t \{ cf(x + c(t-s), s) + cf(x - c(t-s), s) \} \, ds,
\]

\[
\frac{\partial^2 u}{\partial t^2}(x, t) = \frac{1}{2c} \left( cf(x + c(t-t), t) + cf(x - c(t-t), t) \right) + \frac{1}{2c} \int_0^t \left\{ c^2 \frac{df}{dx}(x + c(t-s), s) - c^2 \frac{df}{dx}(x - c(t-s), s) \right\} \, ds
\]

\[
= \frac{1}{2c} \left( cf(x, t) + cf(x, t) \right) + \frac{1}{2c} \int_0^t \left\{ c^2 \frac{df}{dx}(x + c(t-s), s) - c^2 \frac{df}{dx}(x - c(t-s), s) \right\} \, ds
\]

\[
= f(x, t) + \frac{c}{2} \int_0^t \left\{ \frac{df}{dx}(x + c(t-s), s) - \frac{df}{dx}(x - c(t-s), s) \right\} \, ds,
\]

\[
\frac{\partial u}{\partial x}(x, t) = \frac{1}{2c} \int_0^t \left\{ f(x + c(t-s), s) - f(x - c(t-s), s) \right\} \, ds,
\]

\[
\frac{\partial^2 u}{\partial x^2}(x, t) = \frac{1}{2c} \int_0^t \left\{ \frac{df}{dx}(x + c(t-s), s) - \frac{df}{dx}(x - c(t-s), s) \right\} \, ds.
\]

Thus

\[
\frac{\partial^2 u}{\partial t^2}(x, t) - c^2 \frac{\partial^2 u}{\partial x^2}(x, t) = f(x, t) + \frac{c}{2} \int_0^t \left\{ \frac{df}{dx}(x + c(t-s), s) - \frac{df}{dx}(x - c(t-s), s) \right\} \, ds
\]

\[
- \frac{c}{2} \int_0^t \left\{ \frac{df}{dx}(x + c(t-s), s) - \frac{df}{dx}(x - c(t-s), s) \right\} \, ds
\]

\[
= f(x, t),
\]

as desired. It is obvious from the formulas for \( u \) and \( \partial u/\partial t \) that both are identically zero when \( t = 0 \).

3. We wish to show that

\[
H(c(t-s) - |x-y|) = (H((x-y) + c(t-s)) - H((x-y) - c(t-s))) H(t-s)
\]
9.6. GREEN’S FUNCTIONS FOR THE WAVE EQUATION

for all $x, y, t, s \in \mathbb{R}$ such that $c(t - s) \neq |x - y|$. (The two expressions differ when $c(t - s) = |x - y| > 0$, at least when $H$ is defined as in the text: $H(t) = 1$ if $t > 0$ and $H(t) = 0$ if $t \leq 0$.) We have

$$H(c(t - s) - |x - y|) = 1 \Leftrightarrow c(t - s) - |x - y| > 0$$

$$\Leftrightarrow c(t - s) > |x - y|$$

$$\Leftrightarrow -c(t - s) < |x - y| < c(t - s) \text{ and } t - s > 0$$

$$\Leftrightarrow x - y + c(t - s) > 0 \text{ and } x - y - c(t - s) < 0 \text{ and } t - s > 0$$

$$\Rightarrow H(x - y + c(t - s)) = 1 \text{ and } H(x - y - c(t - s)) = 0 \text{ and } H(t - s) = 1$$

$$\Rightarrow (H((x - y) + c(t - s)) - H((x - y) - c(t - s))) H(t - s) = 1.$$

The reasoning can be reversed if we add the condition $c(t - s) \neq |x - y|$. This completes the proof.

5. Let $u$ be the solution to

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = \tilde{f}(x, t), \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x, 0) = 0, \quad -\infty < x < \infty,$$

$$\frac{\partial u}{\partial t}(x, 0) = 0, \quad -\infty < x < \infty,$$

where $\tilde{f}$ is odd ($\tilde{f}(-x, t) = -\tilde{f}(x, t)$). We wish to show that $u$ is also odd. We will do this by defining $v(x, t) = -u(-x, t)$ and proving that $v$ satisfies the same IVP; then, by uniqueness, it will follow that $v = u$, that is, $u(x, t) = -u(-x, t)$, as desired.

We have

$$\frac{\partial v}{\partial t}(x, t) = -\frac{\partial u}{\partial t}(-x, t),$$

$$\frac{\partial^2 v}{\partial t^2}(x, t) = -\frac{\partial^2 u}{\partial t^2}(-x, t),$$

$$\frac{\partial v}{\partial x}(x, t) = \frac{\partial u}{\partial x}(-x, t),$$

$$\frac{\partial^2 v}{\partial x^2}(x, t) = -\frac{\partial^2 u}{\partial x^2}(-x, t).$$

Therefore,

$$\frac{\partial^2 v}{\partial t^2}(x, t) - c^2 \frac{\partial^2 v}{\partial x^2}(x, t) = -\frac{\partial^2 u}{\partial t^2}(-x, t) + c^2 \frac{\partial^2 u}{\partial x^2}(-x, t)$$

$$= -\left(\frac{\partial^2 u}{\partial t^2}(-x, t) - c^2 \frac{\partial^2 u}{\partial x^2}(-x, t)\right)$$

$$= -\tilde{f}(-x, t) = \tilde{f}(x, t)$$

(since $\tilde{f}$ is odd). Thus $v$ satisfies the same wave equation as does $u$. Moreover, it is obvious that $v$ also satisfies the same initial conditions; hence $v = u$ and the proof is complete.

7. We outline the derivation of the Green’s function for

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f(x, t), \quad 0 < x < \ell, \quad t > 0,$$

$$u(x, 0) = 0, \quad 0 < x < \ell,$$

$$\frac{\partial u}{\partial t}(x, 0) = 0, \quad 0 < x < \ell,$$

$$\frac{\partial u}{\partial x}(0, t) = 0, \quad t > 0,$$

$$\frac{\partial u}{\partial x}(\ell, t) = 0, \quad t > 0.$$
We define \( \tilde{f} \) to be the even, periodic extension of \( f \), and define \( \tilde{u} \) to be the solution of
\[
\frac{\partial^2 \tilde{u}}{\partial t^2} - c^2 \frac{\partial^2 \tilde{u}}{\partial x^2} = \tilde{f}(x, t), \quad -\infty < x < \infty, \quad t > 0,
\]
\[
\tilde{u}(x, 0) = 0, \quad -\infty < x < \infty,
\]
\[
\frac{\partial \tilde{u}}{\partial t}(x, 0) = 0, \quad -\infty < x < \infty.
\]

From earlier results, we know that
\[
\tilde{u}(x, t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{2c} \left( H(c(t-s) - |x-y|) \tilde{f}(y, s) dy ds - \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \tilde{f}(y, s) dy ds. \right.
\]

Then
\[
\frac{\partial \tilde{u}}{\partial x}(x, t) = \int_0^t \left\{ \tilde{f}(x + c(t-s), s) - \tilde{f}(x - c(t-s), s) \right\} ds.
\]

It follows that
\[
\frac{\partial \tilde{u}}{\partial x}(x, 0) = \int_0^t \left\{ \tilde{f}(c(t-s), s) - \tilde{f}(-c(t-s), s) \right\} ds,
\]
\[
\frac{\partial \tilde{u}}{\partial x}(x, \ell) = \int_0^t \left\{ \tilde{f}(\ell + c(t-s), s) - \tilde{f}(\ell - c(t-s), s) \right\} ds.
\]

Since \( \tilde{f} \) is even, we have \( \tilde{f}(c(t-s), s) = \tilde{f}(-c(t-s), s) \), and hence
\[
\frac{\partial \tilde{u}}{\partial x}(x, 0) = 0.
\]

Also,
\[
\tilde{f}(\ell - c(t-s), s) = \tilde{f}(-\ell + c(t-s), s) \text{ (since } \tilde{f} \text{ is even)}
\]
\[
= \tilde{f}(\ell + c(t-s), s) \text{ (since } \tilde{f} \text{ is } 2\ell \text{-periodic)}.
\]

Therefore,
\[
\frac{\partial \tilde{u}}{\partial x}(x, \ell) = 0.
\]

It follows that \( \tilde{u} \), restricted to the interval \( 0 < x < \ell \), solves the original IBVP.

The derivation of the Green’s function for the IBVP is almost exactly the same as that given in the text (for Dirichlet conditions). The result is
\[
G(x, t; y, s) = \sum_{n=-\infty}^{\infty} \frac{1}{2c} \left( H(c(t-s) - |x-y-2n\ell|) + H(c(t-s) - |x+y-2n\ell|) \right).
\]
Chapter 10

Sturm-Liouville Eigenvalue Problems

10.2 Properties of the Sturm-Liouville operator

1. The verification that $L$ is symmetric is exactly the same as that given (for the general case) on pages 389–390, except for the treatment of the boundary term. We have

$$\left. -P(x) \frac{du}{dx}(x)v(x) \right|_a^b = -P(b) \frac{du}{dx}(b)v(b) + P(a) \frac{du}{dx}(a)v(a)$$

$$= -P(b) \frac{du}{dx}(b)v(b) \quad \text{(since } v(a) = 0)$$

$$= \frac{\beta_1}{\beta_2} P(b)u(b)v(b) \quad \left(\text{since } \frac{du}{dx}(b) = -\frac{\beta_1}{\beta_2} u(b)\right).$$

Therefore,

$$(Lu, v)_w = \frac{\beta_1}{\beta_2} P(b)u(b)v(b) + \int_a^b P(x) \frac{du}{dx}(x) \frac{dv}{dx}(x) \, dx + \int_a^b R(x)u(x)v(x) \, dx.$$ 

The expression on the right is symmetric in $u$ and $v$, which shows that

$$(u, L(v))_w = (L(v), u)_w = (L(u), v)_w,$$

verifying that $L$ is symmetric with respect to $(\cdot, \cdot)_w$.

3. Consider a Robin condition of the form

$$\alpha_1 u(a) + \kappa \frac{du}{dx}(a) = 0$$

(relative to the interval $[a, b]$). We wish to determine the physically meaningful sign for $\alpha_1$ in the case that this boundary condition models heat flow through the left end of a bar. Recall that the heat flux at $x = a$ (a vector quantity) is given by

$$-\kappa \frac{du}{dx}(a)$$

(the direction of the vector is indicated by the sign). Since the normal direction to the left end of the bar is $-1$, the rate at which heat flows out of the bar at $x = a$ is

$$-\kappa \frac{du}{dx}(a)(-1) = \kappa \frac{du}{dx}(a).$$

Thus consider the equation

$$\kappa \frac{du}{dx}(a) = \gamma(u(a) - T),$$

where $T$ is the temperature of the surroundings. This equation states that the rate at which heat flows out of the bar is proportional to the difference between the temperature at the left end of the bar and the temperature of the bar. The above boundary condition is equivalent to

$$-\gamma u(a) + \kappa \frac{du}{dx}(a) = -\gamma T,$$
which shows that we should take $\alpha_1 = -\gamma$ in the Robin boundary condition. Thus $\alpha_1 < 0$ is the physically meaningful case for a Robin boundary condition at the left endpoint.

At the right endpoint $x = b$, the normal direction is 1, and therefore the rate at which heat flows out of the bar is $\kappa \frac{du}{dx}(b)$.

The same reasoning as above then shows that the boundary condition is $\gamma u(b) + \kappa \frac{du}{dx}(b) = \gamma T$.

Therefore, we should take $\beta_1 = \gamma$, which shows that $\beta_1 > 0$ is the physically meaningful case.

5. The eigenpairs are $\lambda_n = 3 + \left(\frac{n\pi}{\ln 2}\right)^2$, $u_n(x) = \sin \left(\frac{n\pi x}{\ln 2}\right)$, $n = 1, 2, 3, \ldots$.

7. (a) The eigenpairs are $\lambda_n = 4 + \beta + \left(\frac{n\pi}{\ln 2}\right)^2$, $u_n(x) = x^2 \sin \left(\frac{n\pi x}{\ln 2}\right)$, $n = 1, 2, 3, \ldots$.

(b) 0 is an eigenvalue if and only if $\beta = -4 - \left(\frac{n\pi}{\ln 2}\right)^2$ for some positive integer $n$.

(c) There is exactly one negative eigenvalue if and only if $-4 - 4\pi^2 \left(\frac{1}{\ln 2}\right)^2 < \beta < -4 - \pi^2 \left(\frac{1}{\ln 2}\right)^2$.

while there are exactly $k$ negative eigenvalues if and only if $-4 - (k+1)^2 \pi^2 \left(\frac{1}{\ln 2}\right)^2 < \beta < -4 - k^2 \pi^2 \left(\frac{1}{\ln 2}\right)^2$.

9. The eigenpairs are $\lambda_n = 2 + \left(\frac{n\pi}{\ln 2}\right)^2$, $u_n(x) = x^{-1} \left(\frac{n\pi}{\ln 2} \cos \left(\frac{n\pi x}{\ln 2}\right) + \sin \left(\frac{n\pi x}{\ln 2}\right)\right)$, $n = 1, 2, 3, \ldots$.

10.3 Numerical Methods for Sturm-Liouville problems

1. We wish to apply the finite element method to

$$-\frac{d}{dx} \left( P(x) \frac{du}{dx} \right) + R(x) u = \lambda w(x)u, \quad a < x < b,$$

$$u(a) = 0,$$

$$\frac{du}{dx}(b) = 0.$$

Using the usual uniform mesh on the interval $[a, b]$, the finite element space of consists of all piecewise linear functions $v$ (relative to the given mesh) satisfying $v(a) = 0$ (note that the Neumann condition is a natural boundary condition). The standard basis for this space is $\{\phi_1, \ldots, \phi_n\}$ (where each $\phi_j$ is the usual “hat” function). Applying the Galerkin method to the weak form (Equation (10.12) in the text), we obtain $Au = \lambda W u$, where $A$ and $W$ are the $n \times n$ matrices defined by

$$A_{ij} = \int_a^b \left\{ P(x) \frac{d\phi_j}{dx}(x) \frac{d\phi_i}{dx}(x) + R(x)\phi_j(x)\phi_i(x) \right\} dx,$$

$$W_{ij} = \int_a^b w(x)\phi_j(x)\phi_i(x) dx.$$

(These are the same formulas derived in the text, except now $i$ and $j$ vary from 1 to $n$.)
10.4 Examples of Sturm-Liouville problems

1. Let

\[ u(x) = \sum_{i=0}^{n} \alpha_i \phi_i(x) \]

be a continuous piecewise linear function defined on \([0, \ell]\), relative to the usual uniform mesh. Here \(\{\phi_0, \phi_1, \ldots, \phi_n\}\) is the standard basis. The average value of \(u\) on \([0, \ell]\) is then

\[ \frac{1}{\ell} \int_0^\ell u(x) \, dx = \frac{1}{\ell} \sum_{i=0}^{n} \alpha_i \phi_i(x) \, dx = \frac{1}{\ell} \sum_{i=0}^{n} \alpha_i \int_0^\ell \phi_i(x) \, dx. \]

Since the graph of each \(\phi_i, \, i = 1, \ldots, n-1\), forms a triangle of height 1 and base 2 \(h\) (\(h = \ell/n\)), it follows that

\[ \int_0^\ell \phi_i(x) \, dx = h, \quad i = 1, 2, \ldots, n-1; \]

similarly,

\[ \int_0^\ell \phi_0(x) \, dx = \int_0^\ell \phi_n(x) \, dx = \frac{h}{2}, \quad i = 1, 2, \ldots, n-1. \]

Thus the average value of \(u\) on \([0, \ell]\) is

\[ \frac{h}{\ell} \left( \frac{1}{2} \alpha_0 + \alpha_1 + \cdots + \alpha_{n-1} + \frac{1}{2} \alpha_n \right). \]

3. The fundamental frequency is about 3254 Hz.

10.5 Robin boundary conditions

1. The solution to the IBVP is

\[ u(x, t) = \sum_{n=1}^{\infty} a_n(t) \cos \left( \frac{(2n-1)\pi x}{2\ell} \right), \]

where

\[ a_n(t) = c_n e^{-\kappa(2n-1)^2 \pi^2 t / (4\ell^2 \rho c)}, \quad c_n = \frac{40(-1)^{n+1}}{(2n-1)\pi}, \quad n = 1, 2, \ldots. \]

Some snapshots of the solution are shown in Figure 10.1.

3. Consider the following Sturm-Liouville problem:

\[ \frac{d^2 u}{dx^2} - \lambda u, \quad 0 < x < \ell, \]

\[ u(0) = 0, \]

\[ \kappa \frac{d u}{dx}(\ell) + \alpha u(\ell) = 0 \]

where \(\kappa > 0, \alpha < 0\). We can write the second boundary condition as

\[ \frac{d u}{dx}(\ell) - \overline{\alpha} u(\ell) = 0, \]

where \(\overline{\alpha} = -\alpha/\kappa > 0\). The analysis of the eigenvalues turns out to depend on the value of \(\overline{\alpha}\ell\). We do the analysis by considering three cases:
Case 1 \((\lambda > 0)\) The positive eigenvalues are the solutions of the equation

\[
\tan \left( \sqrt{\lambda} \ell \right) = \frac{\sqrt{\lambda}}{\alpha}
\]

or

\[
\tan (s) = \frac{s}{\alpha \ell}, \ s > 0,
\]

where \(s = \sqrt{\lambda} \ell\). There are solutions \(s_k = (2k + 1)\pi/2, \ k = 1, 2, 3, \ldots\). The corresponding eigenpairs are

\[
\lambda_k = \frac{s_k^2}{\ell^2}, \ \psi_k(x) = \sin \left( \frac{s_k x}{\ell} \right), \ k = 1, 2, 3, \ldots.
\]

These eigenpairs exist regardless of the value of \(\alpha \ell\). In addition, if \(\alpha \ell < 1\), then there is an additional solution of \(\tan (s) = s/\alpha \ell\), namely, \(s_0 \in (0, \pi/2)\), with corresponding eigenvalue \(\lambda_0 = s_0^2/\ell^2\) and eigenfunction \(\psi_0(x) = \sin (s_0 x/\ell)\).

Case 2 \((\lambda = 0)\) If \(\alpha \ell = 1\), then \(\lambda_0 = 0\) is an eigenvalue with eigenfunction \(\psi_0(x) = x\). If \(\alpha \ell \neq 1\), then 0 is not an eigenvalue.

Case 3 \((\lambda < 0)\) Negative eigenvalues must satisfy the equation

\[
\tanh \left( \sqrt{-\lambda} \ell \right) = \frac{\sqrt{-\lambda}}{\alpha}
\]

or

\[
\tanh (s) = \frac{s}{\alpha \ell}, \ s > 0,
\]

where \(s = \sqrt{-\lambda} \ell\). If \(\alpha \ell \leq 1\), then this equation has no solution and hence there are no negative eigenvalues. If \(\alpha \ell > 1\), then there is a single solution \(s_0\), leading to a single negative eigenvalue \(\lambda_0 = -s_0^2/\ell^2\) and eigenfunction \(\psi_0(x) = \sinh (s_0 x/\ell)\).

Therefore, when \(\alpha < 0\), there is a sequence of eigenpairs

\[
\lambda_k = \frac{s_k^2}{\ell^2}, \ \psi_k(x) = \sin \left( \frac{s_k x}{\ell} \right), \ k = 1, 2, 3, \ldots.
\]

where

\[
\lambda_k \approx \frac{(2k + 1)^2 \pi^2}{4\ell^2}.
\]

There is also an eigenvalue \(\lambda_0\), which is positive if \(\alpha \ell < 1\) (with eigenfunction \(\psi_0(x) = \sin (\sqrt{\lambda_0} x)\)), zero if \(\alpha \ell = 1\) (with eigenfunction \(\psi_0(x) = x\)), and negative if \(\alpha \ell > 1\) (with eigenfunction \(\psi_0(x) = \sinh (\sqrt{-\lambda_0} x)\)).

10.6 Finite element methods for Robin boundary conditions

1. The finite element formulation of

\[
-\frac{d}{dx} \left( P(x) \frac{du}{dx} \right) + R(x)u = F(x), \ a < x < b,
\]

\[u(a) = 0,
\]

\[\beta \frac{du}{dx}(b) + \alpha u(b) = 0,
\]
where $P(x) > 0$ for all $x \in [a, b]$, $\alpha, \beta > 0$, is nearly the same as for the BVP (10.26) given in the text. The Dirichlet condition at $x = a$ is an essential boundary condition (whereas the Neumann condition in (10.26) is a natural boundary condition). This implies that we only consider finite element functions satisfying the Dirichlet condition, so we use the basis \{ $\phi_1, \phi_2, \ldots, \phi_n$ \} instead of \{ $\phi_0, \phi_1, \ldots, \phi_n$ \}. The weak form of the BVP is unchanged from (10.27), although now

$$
P(a) \frac{du}{dx}(a)v(a)
$$

vanishes because $v(a) = 0$ rather than because $du/dx(a) = 0$. The result is the linear system $Au = f$, where $A$ and $f$ are given by the same formulas as in the text (page 420), except that the entries corresponding to $\phi_0$ are omitted. Thus $A$ is now $n \times n$ and $f$ is now an $n$-vector.

3. The weak form of

$$
- \frac{d}{dx} \left( P(x) \frac{du}{dx} \right) + R(x)u = F(x), \quad a < x < b,
$$

$$
- \alpha_1 u(a) + \alpha_2 \frac{du}{dx}(a) = 0,
$$

$$
\beta_1 u(b) + \beta_2 \frac{du}{dx}(b) = 0
$$

is

$$
P(a) \frac{du}{dx}(a)v(a) - P(b) \frac{du}{dx}(b)v(b) + \int_a^b P(x) \frac{du}{dx}(x) \frac{dv}{dx}(x) dx + \int_a^b R(x)u(x)v(x) dx = \int_a^b F(x)v(x) dx
$$

for all test functions $v$. The boundary conditions imply

$$
\frac{du}{dx}(a) = \frac{\alpha_1}{\alpha_2} u(a), \quad \frac{du}{dx}(b) = -\frac{\beta_1}{\beta_2} u(b),
$$

and therefore the weak form can be written as

$$
\frac{\alpha_1}{\alpha_2} P(a)u(a)v(a) + \frac{\beta_1}{\beta_2} P(b)u(b)v(b) + \int_a^b P(x) \frac{du}{dx}(x) \frac{dv}{dx}(x) dx + \int_a^b R(x)u(x)v(x) dx = \int_a^b F(x)v(x) dx.
$$

Notice that the Robin conditions at the endpoints are natural boundary conditions, and hence no boundary conditions are imposed on the test functions. We therefore take \{ $\phi_0, \phi_1, \ldots, \phi_n$ \} as the basis for the finite element space (using the usual notation), and obtain the linear system $Au = f$, where $A = K + M + G$. The matrices $K$ and $M$ are the same stiffness and mass matrices defined in the text (page 420), and $G \in \mathbb{R}^{(n+1) \times (n+1)}$ is defined by

$$
G_{ij} = \begin{cases} 
\frac{\alpha_1}{\alpha_2} P(a), & i = j = 0, \\
\frac{\beta_1}{\beta_2} P(b)\beta, & i = j = n, \\
0, & \text{otherwise}.
\end{cases}
$$

5. Suppose $u$ satisfies

$$
- \kappa \frac{d^2u}{dx^2}(x) = F(x), \quad 0 < x < \ell,
$$

$$
\frac{du}{dx}(0) = 0,
$$

$$
\kappa \frac{du}{dx}(\ell) + \alpha u(\ell) = 0.
$$

Then

$$
\int_0^\ell F(x) dx = -\kappa \int_0^\ell \frac{d^2u}{dx^2}(x) dx = -\kappa \left( \frac{du}{dx}(\ell) - \frac{du}{dx}(0) \right)
$$

$$
= -\kappa \frac{du}{dx}(\ell) \left( \text{since } \frac{du}{dx}(0) = 0 \right)
$$

$$
= \alpha u(\ell) \quad \text{(applying the Robin condition at } x = \ell).$$
10.7 The theory of Sturm-Liouville problems: an outline

1. Let $L : U \to V$ be a compact linear operator, where $U$, $V$ are normed linear spaces. We wish to show that $L$ is bounded. We will argue by contradiction and assume $L$ is unbounded. Then there exists a sequence $\{u_n\} \subset U$ with $\|u_n\|_U = 1$ for all $n$ and $\|Lu_n\|_V \to \infty$ as $n \to \infty$. (Here we are using the alternate definition of the norm of $L$: $\|L\| = \sup \{\|Lu\|_V : u \in U, \|u\|_U = 1\}$.) Since $L$ is compact and $\{u_n\}$ is bounded in $U$, there must exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{Lu_{n_k}\}$ is convergent in $V$. But $\|Lu_{n_k}\|_V \to \infty$ implies that $\|Lu_{n_k}\|_V \to \infty$, and hence that $\{Lu_{n_k}\}$ is not convergent. This is a contradiction, which shows that the assumption that $L$ is unbounded is not possible. Therefore, $L$ must be bounded, which is what we wanted to prove.

3. Suppose that, for a given $\lambda \in \mathbb{R}$, $u = \phi$, $u = \psi$ both satisfy

$$-\frac{d}{dx} \left( P(x) \frac{du}{dx} \right) + R(x)u = \lambda w(x)u, \quad a < x < b,$$

$$\alpha_1 u(a) + \alpha_2 \frac{du}{dx}(a) = 0,$$

$$\beta_1 u(b) + \beta_2 \frac{du}{dx}(b) = 0,$$

where $P$ and $w$ are assumed to be positive on the interval $[a, b]$. Let

$$W(x) = \left| \begin{array}{cc} \phi(x) & \psi(x) \\ \frac{d\phi}{dx}(x) & \frac{d\psi}{dx}(x) \end{array} \right| = \phi(x) \frac{d\psi}{dx}(x) - \psi(x) \frac{d\phi}{dx}(x)$$

be the Wronskian of $\phi, \psi$. Since $\phi, \psi$ satisfy the linear homogeneous ODE

$$-\frac{d}{dx} \left( P(x) \frac{du}{dx} \right) + (R(x) - \lambda w(x))u = 0$$

on $[a, b]$, we can show that $\{\phi, \psi\}$ is linearly independent by proving that $W(x) = 0$ for any given $x \in [a, b]$. We have

$$\alpha_1 \phi(a) + \alpha_2 \frac{d\phi}{dx}(a) = 0 \Rightarrow \frac{d\phi}{dx}(a) = -\frac{\alpha_1}{\alpha_2} \phi(a),$$

and similarly

$$\frac{d\psi}{dx}(a) = -\frac{\alpha_1}{\alpha_2} \psi(a).$$

Therefore,

$$W(a) = \phi(a) \frac{d\psi}{dx}(a) - \frac{d\phi}{dx}(a) \psi(a) = -\frac{\alpha_1}{\alpha_2} \phi(a) \psi(a) + \frac{\alpha_1}{\alpha_2} \phi(a) \psi(a) = 0.$$

This shows that $\{\phi, \psi\}$ is linearly dependent, as desired.
Chapter 11

Problems in Multiple Spatial Dimensions

11.1 Physical models in two or three spatial dimensions

1. Let \( \Omega \subset \mathbb{R}^2 \) be the rectangular domain
   \[ \Omega = \{ x \in \mathbb{R}^2 : a < x_1 < b, c < x_2 < d \} , \]
   and let \( F : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be a smooth vector field. Then
   \[ \int_{\Omega} \nabla \cdot F = \int_{a}^{b} \int_{c}^{d} \left\{ \frac{\partial F_1}{\partial x_1} (x_1, x_2) + \frac{\partial F_2}{\partial x_2} (x_1, x_2) \right\} \, dx_2 \, dx_1 \]
   \[ = \int_{a}^{b} \int_{c}^{d} \frac{\partial F_1}{\partial x_1} (x_1, x_2) \, dx_2 \, dx_1 + \int_{a}^{b} \int_{c}^{d} \frac{\partial F_2}{\partial x_2} (x_1, x_2) \, dx_2 \, dx_1 \]
   \[ = \int_{c}^{d} \int_{a}^{b} \frac{\partial F_1}{\partial x_1} (x_1, x_2) \, dx_1 \, dx_2 + \int_{a}^{b} \int_{c}^{d} \frac{\partial F_2}{\partial x_2} (x_1, x_2) \, dx_2 \, dx_1 \]
   \[ = \int_{c}^{d} \{ F_1 (b, x_2) - F_1 (a, x_2) \} \, dx_2 + \int_{a}^{b} \{ F_2 (x_1, d) - F_2 (x_1, c) \} \, dx_1 \]
   \[ = \int_{c}^{d} F_1 (b, x_2) \, dx_2 - \int_{c}^{d} F_1 (a, x_2) \, dx_2 + \int_{a}^{b} F_2 (x_1, d) \, dx_1 - \int_{a}^{b} F_2 (x_1, c) \, dx_1 \]
   \[ = \int_{\Gamma_2} F \cdot n + \int_{\Gamma_4} F \cdot n + \int_{\Gamma_3} F \cdot n + \int_{\Gamma_1} F \cdot n = \int_{\partial \Omega} F \cdot n , \]
   as desired. Here \( \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \) denote the four sides of \( \partial \Omega \), labeled as in Figure 11.3 on page 452 of the text.

3. We have \( \nabla \cdot F(x) = 1 \), and therefore
   \[ \int_{\Omega} \nabla \cdot F = \text{area}(\Omega) = \pi . \]
   On the other hand, on \( \partial \Omega \), \( n = x \) and therefore \( F \cdot n = 2x_1x_2 + x_2^2 \). In polar coordinates, \( F \cdot n = \sin (2\theta) + \sin^2 (\theta) \), and hence
   \[ \int_{\partial \Omega} F \cdot n = \int_{0}^{2\pi} \{ \sin (2\theta) + \sin^2 (\theta) \} \, d\theta = \pi . \]
   This verifies the divergence theorem for this domain and vector field.

5. Let \( L : C^2(\Omega) \rightarrow C(\Omega) \) be defined by \( Lu = -\nabla \cdot (k(x)\nabla u) \). By the product rule,
   \[ \nabla \cdot (k(\nabla u)v) = k\nabla u \cdot \nabla v + \nabla \cdot (k\nabla u)v , \]
   therefore,
   \[ -\int_{\Omega} \nabla \cdot (k\nabla u)v = \int_{\Omega} k\nabla u \cdot \nabla v - \int_{\Omega} \nabla \cdot (k\nabla u)v . \]
By the divergence theorem,
\[ \int_{\Omega} \nabla \cdot (k(\nabla u)v) = \int_{\partial \Omega} k(\nabla u)v \cdot n = \int_{\partial \Omega} k \frac{\partial u}{\partial n} v. \]
The boundary term disappears if \( u \) and \( v \) both satisfy Dirichlet condition (since then \( v = 0 \) on \( \partial \Omega \)) and also if \( u \) and \( v \) both satisfy Neumann conditions (since then \( \partial u/\partial n = 0 \) on \( \partial \Omega \)). Thus, in either case,
\[ (Lu, v) = -\int_{\Omega} \nabla \cdot (k(\nabla u)v) = \int_{\Omega} k \nabla u \cdot \nabla v. \]
Since the right-hand side is symmetric in \( u \) and \( v \), so is the left-hand side; that is, \((Lu, v) = (Lv, u) = (u, Lv)\), and we see that \( L \) is a symmetric operator.

7. We have
\[ \nabla \cdot F(u(x)) = \frac{\partial}{\partial x_1} (F_1(u(x))) + \frac{\partial}{\partial x_2} (F_2(u(x))) + \frac{\partial}{\partial x_3} (F_3(u(x))) \]
\[ = F'_1(u(x)) \frac{\partial u}{\partial x_1} (x) + F'_2(u(x)) \frac{\partial u}{\partial x_2} (x) + F'_3(u(x)) \frac{\partial u}{\partial x_3} (x) \]
\[ = F'(u(x)) \cdot \nabla u(x). \]

9. Using the product rule for scalar functions, we obtain
\[ \nabla \cdot (v \nabla u) = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left[ v \frac{\partial u}{\partial x_i} \right] = \sum_{i=1}^{3} \left\{ \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_i} + v \frac{\partial^2 u}{\partial x_i^2} \right\} = \sum_{i=1}^{3} \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_i} + v \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2} \]
\[ = \nabla v \cdot \nabla u + v \Delta u. \]

11. Suppose \( u, v \in C^2_m(\Omega) \). Then
\[ (L_m u, v) = -\int_{\Omega} v \Delta u = \int_{\Omega} \nabla u \cdot \nabla v - \int_{\partial \Omega} v \frac{\partial u}{\partial n} \] (Green’s first identity)
\[ = \int_{\Omega} \nabla u \cdot \nabla v - \int_{\partial \Omega} u \frac{\partial v}{\partial n} \] (Green’s first identity)
\[ = -\int_{\Omega} u \Delta v = (u, L_m v). \]

The boundary terms vanish because the product that forms the integrand is zero over the entire boundary. For example,
\[ \int_{\partial \Omega} v \frac{\partial u}{\partial n} = 0 \]
since \( v = 0 \) on \( \Gamma_1 \) and \( \partial u/\partial n = 0 \) on \( \Gamma_2 \).

### 11.2 Fourier series on a rectangular domain

1. (a)
\[ c_{mn} = -\frac{4 (5 + 7(-1)^m) (-1 + (-1)^n)}{m^3 n^3 \pi^6} \]
(b) The graphs of the error are given in Figure 11.1.

3. The solution is given by
\[ u(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} \lambda_{mn} \sin (m \pi x_1) \sin (n \pi x_2), \]
where the \( c_{mn} \) are the coefficients from Exercise 1 and
\[ \lambda_{mn} = (m^2 + n^2) \pi^2. \]
11.2. FOURIER SERIES ON A RECTANGULAR DOMAIN

Figure 11.1: The error in approximating \( f(x,y) \) by the first 4 terms (top) and the first 25 terms (bottom) of the double Fourier sine series. (See Exercise 11.2.1.)

5. (a) The IBVP is

\[
\rho \frac{\partial u}{\partial t} - \kappa \Delta u = 0.02, \ x \in \Omega, \ t > 0,
\]

\[
u(x,0) = 5, \ x \in \Omega,
\]

\[
u(x,t) = 0, \ x \in \partial \Omega, \ t > 0.
\]

The domain \( \Omega \) is the rectangle \( \{ x \in \mathbb{R}^2 : 0 < x_1 < 50, 0 < x_2 < 50 \} \).

(b) The solution is

\[
u(x,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn}(t) \sin \left( \frac{m\pi x_1}{50} \right) \sin \left( \frac{n\pi x_2}{50} \right),
\]

where

\[
a_{mn}(t) = \left( b_{mn} - \frac{c_{mn}}{\kappa \lambda_{mn}} \right) e^{-\kappa \lambda_{mn} t / (\rho c)} + \frac{c_{mn}}{\kappa \lambda_{mn}},
\]

\[
b_{mn} = \frac{20 (-1 + (-1)^m) (-1 + (-1)^n)}{m \pi^2},
\]

\[
c_{mn} = \frac{2 (-1 + (-1)^m) (-1 + (-1)^n)}{25 m n \pi^2},
\]

\[
\lambda_{mn} = \frac{(m^2 + n^2) \pi^2}{2500}.
\]

(c) The steady-state temperature \( u_s(x) \) satisfies the BVP

\[
-\kappa \Delta u = 0.02, \ x \in \Omega,
\]

\[
u(x) = 0, \ x \in \partial \Omega.
\]

The solution is

\[
u_s(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} \sin \left( \frac{m\pi x_1}{50} \right) \sin \left( \frac{n\pi x_2}{50} \right).
\]

(d) The maximum difference between the temperature after 10 minutes and the steady-state temperature is about 1 degree. The difference is graphed in Figure 11.2.

7. The minimum temperature in the plate reaches 4 degrees Celsius after 825 seconds.

9. The leading edge of the wave is initially \( \frac{2}{5} \) units from the boundary, and the wave travels at a speed of \( 261 \sqrt{2} \) units per second. Therefore, the wave reaches the boundary after \( \sqrt{2}/1305 \approx 0.00108 \) seconds. Figure 11.6 from the text shows the wave about to reach the boundary after \( 10^{-3} \) seconds.
Figure 11.2: The difference between the temperature in the plate after 10 minutes and the steady-state temperature. (See Exercise 11.2.5.)

11. The difficult task is to compute the Fourier coefficients of the initial displacement $\psi$. If we write

$$\psi(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \sin (m\pi x_1) \sin (n\pi x_2),$$

then we find that

$$b_{mn} = \begin{cases} 0, & m \neq n, \\ \frac{\sin^2(m\pi/2)}{1250m^2n^2}, & m = n. \end{cases}$$

We then have that

$$u(x, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn}(t) \sin (m\pi x_1) \sin (n\pi x_2),$$

where $a_{mn}(t)$ satisfies

$$\frac{d^2a_{mn}}{dt^2} + \lambda_{mn}c^2a_{mn} = 0,$$
$$a_{mn}(0) = b_{mn},$$
$$\frac{da_{mn}}{dt}(0) = 0.$$

The result is

$$a_{mn}(t) = \begin{cases} 0, & m \neq n, \\ b_{mn} \cos (c\sqrt{\lambda_{mn}}t), & m = n. \end{cases}$$

Thus the solution is

$$u(x, t) = \sum_{m=1}^{\infty} b_{mn} \cos (c\sqrt{\lambda_{mn}}t) \sin (m\pi x_1) \sin (n\pi x_2).$$

Four snapshots of $u$ are shown in Figure 11.3.

13. The solution is

$$u(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4 \sin (m\pi/2) \sin (n\pi/2)}{(m^2 + n^2)\pi^2} \sin (m\pi x_1) \sin (n\pi x_2).$$

The graph of $u$ is shown in Figure 11.4.

15. (a) The IBVP is

$$\rho c \frac{\partial u}{\partial t} - \kappa \Delta u = 0.02, \ x \in \Omega, \ t > 0,$$
$$u(x, 0) = 5, \ x \in \Omega,$$
$$u(x, t) = 0, \ x \in \Gamma_1 \cup \Gamma_4, \ t > 0,$$
$$\frac{\partial u}{\partial n}(x, t) = 0, \ x \in \Gamma_2 \cup \Gamma_3, \ t > 0.$$

The domain $\Omega$ is the rectangle

$$\{x \in \mathbb{R}^2 : 0 < x_1 < 50, \ 0 < x_2 < 50\}.$$
11.2. FOURIER SERIES ON A RECTANGULAR DOMAIN

Figure 11.3: Four snapshots of the solution to Exercise 11.2.11: $t = 0$ (top left), $t = T/8$ (top right), $t = T/2$ (bottom left), $t = 5T/8$ (bottom right). The solution is periodic with period $T$.

(b) The solution is

$$u(x, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn}(t) \sin \left( \frac{(2m-1)\pi x_1}{100} \right) \sin \left( \frac{(2n-1)\pi x_2}{100} \right),$$

where

$$a_{mn}(t) = \left( b_{mn} - \frac{c_{mn}}{\kappa \lambda_{mn}} \right) e^{-\kappa \lambda_{mn} t / (\rho c)} + \frac{c_{mn}}{\kappa \lambda_{mn}},$$

$$b_{mn} = \frac{80}{(2m-1)(2n-1)\pi^2},$$

$$c_{mn} = \frac{8}{25(2m-1)(2n-1)\pi^2},$$

$$\lambda_{mn} = \frac{(2m-1)^2 + (2n-1)^2}{10000}\pi^2.$$

(c) The steady-state temperature $u_s(x)$ satisfies the BVP

$$-\kappa \Delta u = 0.02, \; x \in \Omega,$$

$$u(x) = 0, \; x \in \Gamma_1 \cup \Gamma_4,$$

$$\frac{\partial u}{\partial n}(x) = 0, \; x \in \Gamma_2 \cup \Gamma_3.$$

The solution is

$$u_s(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{c_{mn}}{\kappa \lambda_{mn}} \sin \left( \frac{(2m-1)\pi x_1}{100} \right) \sin \left( \frac{(2n-1)\pi x_2}{100} \right).$$

Figure 11.4: The solution to Exercise 11.2.13.
(d) After 10 minutes, the temperature $u$ is not very close to the steady-state temperature; the difference $u(x, y, 600) - u_s(x, y)$ is graphed in Figure 11.5. (Note: The temperature variation in this problem may be outside the range in which the linear model is valid, so these results may be regarded with some skepticism.)

Figure 11.5: The difference between the temperature in the plate after 10 minutes and the steady-state temperature. (See Exercise 11.2.15.)

11.3 Fourier series on a disk

1. The next three coefficients in the series for $g$ are

$$ c_{40} \doteq -0.0209908, \quad c_{50} \doteq 0.0116362, \quad c_{60} \doteq -0.0072147. $$

3. The solution is

$$ u(r, \theta) = \sum_{m=1}^{\infty} a_{m0} J_0(s_{0m} r), $$

where

$$ a_{m0} = \frac{c_{m0}}{s_{0m}^2}, $$

the $c_{m0}$ are the coefficients of $f(r, \theta) = 1 - r$, and the $s_{0m}$ are the positive roots of $J_0$. A direct calculation shows that

$$ a_{10} \doteq 0.226324, $$
$$ a_{20} \doteq -0.0157747, $$
$$ a_{30} \doteq 0.00386078, $$
$$ a_{40} \doteq -0.00148156, $$
$$ a_{50} \doteq 0.000716858, $$
$$ a_{60} \doteq -0.000399554. $$

The solution (approximated by six terms of the series) is graphed in Figure 11.6.

5. The solution is

$$ u(r, \theta) = \sum_{m=1}^{\infty} a_{m0} J_0(s_{0m} r), $$

where

$$ a_{m0} = \frac{c_{m0}}{s_{0m}^2}, $$
11.3. FOURIER SERIES ON A DISK

Figure 11.6: The approximation to solution \( u \), computed with six terms of the (generalized) Fourier series (see Exercise 11.3.3).

\[
\begin{align*}
\phi(r, \theta) &= \sum_{m=1}^{\infty} c_{m0} J_0(\alpha_{m0} r) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (c_{mn} \cos(n\theta) + d_{mn} \sin(n\theta)) J_n(\alpha_{mn} r), \\
u(r, \theta, t) &= \sum_{m=1}^{\infty} a_{m0}(t) J_0(\alpha_{m0} r) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (a_{mn}(t) \cos(n\theta) + b_{mn}(t) \sin(n\theta)) J_n(\alpha_{mn} r),
\end{align*}
\]

where the \( c_{mn} \) and \( d_{mn} \) are known, and

\[
\begin{align*}
a_{mn}(t) &= c_{mn} e^{-\kappa \lambda_{mn} t/(\mu c)}, \quad m = 1, 2, 3, \ldots, \ n = 0, 1, 2, \ldots, \\
b_{mn}(t) &= d_{mn} e^{-\kappa \lambda_{mn} t/(\mu c)}, \quad m, n = 1, 2, 3, \ldots.
\end{align*}
\]

The solution (approximated by six terms of the series) is graphed in Figure 11.7.

Figure 11.7: The approximation to solution \( u \), computed with six terms of the (generalized) Fourier series (see Exercise 11.3.5).
However, since $\phi(r, \theta) = r(1-r) \cos(\theta)/5$ (a function of $r$ times $\cos(\theta)$), all of the coefficients of $\phi$ are zero except for $c_{m1}, m = 1, 2, 3, \ldots$. Therefore,

$$u(r, \theta, t) = \sum_{m=1}^{\infty} a_{m1}(t) \cos(\theta) J_1(\alpha_{m1} r),$$

with $a_{m1}(t)$ given above. After 30 seconds, the temperature distribution can be approximated accurately using a single eigenfunction (corresponding to the largest eigenvalue, $\lambda_{11}$), so we need only

$$c_{11} \doteq 9.04433.$$

The temperature distribution after 30 seconds is graphed in Figure 11.8.

Figure 11.8: The approximation to solution $u$ at $t = 30$, computed with one term of the (generalized) Fourier series (see Exercise 11.3.7).

9. A circular drum of radius $A$ has area $\pi A^2$, the same area as a square drum of side length $\sqrt{\pi} A$. The fundamental frequency of such a square drum is

$$c \sqrt{\frac{\pi^2}{(\sqrt{\pi} A)^2} + \frac{\pi^2}{(\sqrt{\pi} A)^2}} = \frac{c}{A \sqrt{2\pi}} \doteq 0.398942 \frac{c}{A}.$$

Comparing to Example 11.9, we see that the circular drum sounds a lower frequency than a square drum of equal area.

11. $s_{41} \doteq 7.588342434503804, s_{42} \doteq 11.06470948850118, s_{43} \doteq 14.37253667161759, s_{44} \doteq 17.61596604980483$

### 11.4 Finite elements in two dimensions

1. The mesh for this problem is shown in Figure 11.9, in which the free nodes are labeled. The stiffness matrix is

$$K \doteq \begin{bmatrix}
4.4815 & -1.759 & -1.759 & 0.0000 \\
-1.759 & 4.9259 & 0.0000 & -1.3426 \\
-1.759 & 0.0000 & 4.9259 & -1.3426 \\
0.0000 & -1.3426 & -1.3426 & 5.8148
\end{bmatrix},$$

while the load vector is

$$F \doteq \begin{bmatrix}
0.11111 \\
0.11111 \\
0.11111 \\
0.11111
\end{bmatrix}.$$

The resulting weights for the finite element approximation are given by

$$u = K^{-1}F \doteq \begin{bmatrix}
0.048398 \\
0.044979 \\
0.044979 \\
0.039879
\end{bmatrix}.$$
11.4. FINITE ELEMENTS IN TWO DIMENSIONS

![Figure 11.9: The mesh for Exercise 11.4.1.](image)

3. The mesh for this problem is shown in Figure 11.10, in which the free nodes are labeled. The stiffness matrix is $16 \times 16$ and neither it nor the load vector will be reproduced here. The matrix is singular, as is expected for a Neumann problem. The unique solution to $KU = F$ with its last component equal to zero is

$$
\begin{bmatrix}
-0.071138 \\
-0.047315 \\
-0.012656 \\
0.001458 \\
-0.067934 \\
-0.047349 \\
-0.014145 \\
0.001566 \\
-0.065432 \\
-0.046198 \\
-0.015160 \\
-0.000329 \\
-0.062914 \\
-0.016065 \\
0.000000
\end{bmatrix}.
$$

![Figure 11.10: The mesh for Exercise 11.4.3.](image)

5. Let $V_n$ be the space of continuous piecewise linear functions relative to a given triangulation of the domain $\Omega$, and let $\{\phi_1, \ldots, \phi_n\}$ be the standard basis for $V_n$. Suppose that $u : \Omega \to \mathbb{R}$ belongs to $L^2(\Omega)$. We wish to find $v \in V_n$ that minimizes $\|w - u\|_{L^2(\Omega)}$ over all $w \in V_n$. We can write $v = \sum_{i=1}^{n} \alpha_i \phi_i$ and use the projection theorem: $v$ must satisfy $(u - v, w)_{L^2(\Omega)} = 0$ for all $w \in V_n$, which is equivalent to $(u - v, \phi_i)_{L^2(\Omega)} = 0$ for all $i = 1, 2, \ldots, n$. 
CHAPTER 11. PROBLEMS IN MULTIPLE SPATIAL DIMENSIONS

We obtain the following equations:

\[(v, \phi_i) = (u, \phi_i), \ i = 1, 2, \ldots, n\]

\[\Rightarrow \sum_{j=1}^{n} \alpha_j \phi_j, \phi_i = (u, \phi_i), \ i = 1, 2, \ldots, n\]

\[\Rightarrow \sum_{j=1}^{n} \alpha_j (\phi_j, \phi_i) = (u, \phi_i), \ i = 1, 2, \ldots, n.\]

This last system of equations is equivalent to \(M \alpha = f\), where \(M\) is the usual mass matrix,

\[M_{ij} = \int_{\Omega} \phi_j \phi_i, \ i, j = 1, 2, \ldots, n,\]

and \(f\) is the vector defined by

\[f_i = \int_{\Omega} v \phi_i, \ i = 1, 2, \ldots, n.\]

7. (a) A direct calculation shows that the inhomogeneous Dirichlet problem

\[-\nabla \cdot (k(x) \nabla u) = f(x), \ x \in \Omega,
\]

\[u(x) = g(x), \ x \in \partial \Omega,\]

has weak form

\[u \in G + V, \ a(u, v) = (f, v) \text{ for all } v \in V = C^2_0(\Omega),\]

where \(G\) is any function satisfying the condition \(G(x) = g(x) \text{ for } x \in \partial \Omega,\) and

\[G + V = \{G + v : v \in V\} .\]

Substituting \(u = G + w, \) where \(w \in V, \) into the weak form yields

\[w \in V, \ a(w, v) = (f, v) - a(G, v) \text{ for all } v \in V.\]

In the Galerkin method, we replace \(V\) by a finite-dimensional subspace \(V_n\) and solve

\[w \in V_n, \ a(w, v) = (f, v) - a(G, v) \text{ for all } v \in V_n.\]

When using piecewise linear finite elements, we can satisfy the boundary conditions approximately by taking \(G\) to be a continuous piecewise linear function whose values at the boundary nodes agree with the given boundary function \(g.\) For simplicity, we take \(G\) to be zero at the interior nodes. The resulting load vector is then given by

\[f_i = (f, \phi_i) - a(G, \phi_i), \ i = 1, 2, 3, \ldots, n.\]

Since \(G\) is zero on interior nodes, the quantity \(a(G, \phi_i)\) is nonzero only if (free) node \(i\) belongs to a triangle adjacent to the boundary.

(b) The regular triangulation of the unit square having 18 triangles has only four interior (free) nodes, and each one belongs to triangles adjacent to the boundary. This means that every entry in the load vector is modified (which is not the typical case). Since \(f = 0,\) the load vector \(f\) is defined by

\[f_i = -a(G, \phi_i), \ i = 1, 2, 3, 4.\]

The load vector is

\[f = \begin{bmatrix} 0.22222222222222 \ 1.55555555555556 \ 0.22222222222222 \ 1.55555555555556 \end{bmatrix},\]

while the solution to \(Ku = f\) is

\[u = \begin{bmatrix} 0.27777777777778 \ 0.61111111111111 \ 0.27777777777778 \ 0.61111111111111 \end{bmatrix}.\]
9. (a) Consider the BVP

\[-\nabla \cdot (k(x)\nabla u) = f(x), \ \text{in} \ \Omega,\]
\[\frac{\partial u}{\partial n}(x) = h(x), \ x \in \partial \Omega.\]  

Multiplying the PDE by a test function \(v \in \tilde{V} = C^2(\Omega)\) and applying Green’s first identity yields

\[\int_{\Omega} k\nabla u \cdot \nabla v = \int_{\Omega} fv + \int_{\partial \Omega} kh v \quad \text{for all} \quad v \in \tilde{V}.
\]

We will apply Galerkin’s method with \(\tilde{V}_m\) equal to the space of all continuous piecewise linear function relative to a given triangulation \(T\). We label the standard basis of \(\tilde{V}_m\) as \(\phi_1, \phi_2, \ldots, \phi_n\), and the nodes as \(z_1, z_2, \ldots, z_n\). (Every node in the mesh is now free.) Thus \(\tilde{V}_m = \text{span}\{\phi_1, \phi_2, \ldots, \phi_n\}\).

We then define the Galerkin approximation \(w_n\) by

\[w_n \in \tilde{V}_m, \quad a(w_n, \phi_i) = (f, \phi_i) + \int_{\partial \Omega} kh \phi_i, \ i = 1, 2, \ldots, m.
\]

Writing \(w_n = \sum_{j=1}^m \beta_j \phi_j\) and

\[w = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix},
\]

we obtain the system \(\tilde{K}w = \tilde{F}\). The stiffness matrix \(\tilde{K} \in \mathbb{R}^{m \times m}\) is given by

\[\tilde{K}_{ij} = a(\phi_j, \phi_i), \ i, j = 1, 2, \ldots, n,
\]

and the load vector \(\tilde{F} \in \mathbb{R}^m\) by

\[\tilde{F}_i = (f, \phi_i) + \int_{\partial \Omega} kh \phi_i, \ i = 1, 2, \ldots, n.
\]

The boundary integral in the expression for \(\tilde{F}_i\) is zero unless \(z_i\) is a boundary node.

(b) The compatibility condition for (11.1) is determined by the following calculation:

\[\int_{\Omega} f = -\int_{\Omega} \nabla \cdot (k \nabla u) = -\int_{\partial \Omega} k \frac{\partial u}{\partial n} = -\int_{\partial \Omega} kh.
\]

(c) Now consider (11.1) with

\[k(x) = 1, \ f(x) = x_1 x_2 + \frac{3}{4}, \ h(x) = \frac{3x_1^2}{5},
\]

where \(\Omega\) is the unit square. We will produce the finite element solution using the regular grid with 18 triangles (16 nodes). The stiffness matrix \(\tilde{K} \in \mathbb{R}^{16 \times 16}\) is singular, as should be expected, and there are infinitely many solutions. The unique solution \(u\) with last component equal to zero is

\[
\begin{bmatrix}
0.3622 \\
0.2984 \\
0.1344 \\
-0.0843 \\
0.3728 \\
0.3302 \\
0.2081 \\
0.0255 \\
0.3803 \\
0.3447 \\
0.2334 \\
0.0591 \\
0.3665 \\
0.3257 \\
0.1878 \\
0.0000
\end{bmatrix}.
\]
11.5 The free-space Green’s function for the Laplacian

1. Following the hint and using the trigonometric identities

\[ \cos(a - b) = \cos(a) \cos(b) + \sin(a) \sin(b), \quad \sin(a - b) = \sin(a) \cos(b) - \cos(a) \sin(b), \]

we obtain

\[ y_1 = r \cos(\theta - \alpha) = r \cos(\theta) \cos(\alpha) + r \sin(\theta) \sin(\alpha) = x_1 \cos(\alpha) + x_2 \sin(\alpha), \]
\[ y_2 = r \sin(\theta - \alpha) = r \sin(\theta) \cos(\alpha) - r \cos(\theta) \sin(\alpha) = -x_1 \sin(\alpha) + x_2 \cos(\alpha), \]

as desired.

3. We have

\[ - \frac{d^2 \psi}{dr^2} - \frac{1}{r} \frac{d\psi}{dr} = \phi(r) \Rightarrow - \frac{d^2 \psi}{dr^2} - \frac{d\psi}{dr} = r \phi(r) \]
\[ \Rightarrow - \frac{d}{dr} \left( r \frac{d\psi}{dr} \right) = r \phi(r). \]

This last ODE can be solved by integrating twice to yield the general solution

\[ \psi(r) = c_1 + c_2 \ln(r) - \int_0^r \ln \left( \frac{r}{s} \right) \phi(s) s \, ds. \]

5. Let \( V \) be the space of all smooth functions defined on \( \mathbb{R}^2 \) and having compact support. We wish to show that the variational problem

\[ \int_{\mathbb{R}^2} \nabla u(x) \cdot \nabla v(x) \, dx = v(y) \text{ for all } v \in V \]

is equivalent to

\[ \int_0^{2\pi} \int_0^\infty \left( \frac{\partial u}{\partial r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial u}{\partial \theta} \frac{\partial v}{\partial \theta} \right) r \, dr \, d\theta = v(y) \text{ for all } v \in V \]

in polar coordinates.

(a) We first convert

\[ \int_{\mathbb{R}^2} \nabla u(x) \cdot \nabla v(x) \, dx \]

to polar coordinates, where \( y \in \mathbb{R}^2 \) is fixed and represents the origin for the polar coordinates. By the chain rule, we have

\[ \frac{\partial u}{\partial x_1} = \cos(\theta) \frac{\partial u}{\partial r} - \sin(\theta) \frac{\partial u}{\partial \theta}, \quad \frac{\partial u}{\partial x_2} = \sin(\theta) \frac{\partial u}{\partial r} + \frac{\cos(\theta)}{r} \frac{\partial u}{\partial \theta} \]

(cf. the derivation in Section 11.3.1) and similarly for \( v \). Therefore,

\[ \nabla u \cdot \nabla v = \left( \cos(\theta) \frac{\partial u}{\partial r} - \sin(\theta) \frac{\partial u}{\partial \theta} \right) \left( \cos(\theta) \frac{\partial v}{\partial r} - \sin(\theta) \frac{\partial v}{\partial \theta} \right) + \]
\[ \left( \sin(\theta) \frac{\partial u}{\partial r} + \frac{\cos(\theta)}{r} \frac{\partial u}{\partial \theta} \right) \left( \sin(\theta) \frac{\partial v}{\partial r} + \frac{\cos(\theta)}{r} \frac{\partial v}{\partial \theta} \right) \]
\[ = \cos^2(\theta) \frac{\partial u}{\partial r} \frac{\partial v}{\partial r} - \sin^2(\theta) \cos(\theta) \frac{\partial u}{\partial r} \frac{\partial v}{\partial \theta} - \sin^2(\theta) \cos(\theta) \frac{\partial u}{\partial \theta} \frac{\partial v}{\partial r} + \frac{\sin^2(\theta)}{r^2} \frac{\partial u}{\partial \theta} \frac{\partial v}{\partial \theta} + \]
\[ \sin^2(\theta) \frac{\partial u}{\partial r} \frac{\partial v}{\partial r} + \sin^2(\theta) \cos(\theta) \frac{\partial u}{\partial r} \frac{\partial v}{\partial \theta} + \frac{\sin^2(\theta)}{r^2} \frac{\partial u}{\partial \theta} \frac{\partial v}{\partial \theta} + \cos^2(\theta) \frac{\partial u}{\partial \theta} \frac{\partial v}{\partial \theta} \]
\[ = \frac{\partial u}{\partial r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial u}{\partial \theta} \frac{\partial v}{\partial \theta}. \]
11.5. The Free-Space Green’s Function for the Laplacian

It follows that
\[ \int_{\mathbb{R}^2} \nabla u(x) \cdot \nabla v(x) \, dx = \int_{0}^{2\pi} \int_{0}^{\infty} \left( \frac{\partial u}{\partial r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial u}{\partial \theta} \frac{\partial v}{\partial \theta} \right) v \, r \, dr \, d\theta, \]
as desired.

(b) We can derive the same result by starting with the PDE \(-\Delta u = \delta(x - y)\) in polar coordinates, multiplying by a test function, and integrating. The right-hand side becomes
\[ \int_{\mathbb{R}^2} \delta(x - y) v(x) \, dx = v(y). \]

On the left side, we have
\[
\begin{align*}
&\int_{0}^{2\pi} \int_{0}^{\infty} \left( \frac{\partial^2 u}{\partial r^2} - \frac{1}{r} \frac{\partial u}{\partial r} - \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right) v \, r \, dr \, d\theta \\
&= -\int_{0}^{2\pi} \int_{0}^{\infty} \frac{\partial^2 u}{\partial r^2} v \, r \, dr \, d\theta - \int_{0}^{2\pi} \int_{0}^{\infty} \frac{\partial u}{\partial r} \frac{\partial v}{\partial r} r \, dr \, d\theta - \int_{0}^{2\pi} \int_{0}^{\infty} \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} v \, dr \, d\theta \\
&= \int_{0}^{2\pi} \left\{ -\frac{\partial u}{\partial r} v \bigg|_{0}^{\infty} + \int_{0}^{\infty} \frac{\partial u}{\partial r} \frac{\partial v}{\partial r} r \, dr + \int_{0}^{\infty} \frac{\partial u}{\partial r} \frac{\partial v}{\partial r} r \, dr \right\} d\theta \\
&= \int_{0}^{2\pi} \left\{ \frac{\partial u}{\partial r} v \bigg|_{0}^{\infty} + \frac{1}{r^2} \frac{\partial u}{\partial \theta} \frac{\partial v}{\partial \theta} r \, dr \right\} d\theta \\
&= \int_{0}^{2\pi} \left\{ \frac{\partial u}{\partial r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial u}{\partial \theta} \frac{\partial v}{\partial \theta} \right\} r \, dr \, d\theta.
\end{align*}
\]

Once again, we have derived the desired result.

7. The derivation is long and tedious. As of the time of this writing, it can be found at planetmath.org/encyclopedia/DerivationOfTheLaplacianFromRectangularToSphericalCoordinates.html.

9. Let \( g(\rho) = c_1/\rho \). Then
\[ \frac{\partial g}{\partial \rho} (\rho) = -\frac{c_1}{\rho^2} \]
and therefore
\[
\begin{align*}
&\int_{0}^{2\pi} \int_{0}^{\infty} \frac{\partial g}{\partial \rho} \frac{\partial v}{\partial \rho} \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta = -c_1 \int_{0}^{2\pi} \int_{0}^{\infty} \frac{1}{\rho^2} \frac{\partial v}{\partial \rho} \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta \\
&= -c_1 \int_{0}^{2\pi} \int_{0}^{\infty} \frac{\partial v}{\partial \rho} \sin(\phi) \, d\rho \, d\phi \, d\theta \\
&= -c_1 \int_{0}^{2\pi} \int_{0}^{\infty} \sin(\phi) \left( \int_{0}^{\infty} \frac{\partial v}{\partial \rho} \, d\rho \right) \, d\phi \, d\theta \\
&= -c_1 \int_{0}^{2\pi} \int_{0}^{\infty} \sin(\phi) (-v(y)) \, d\phi \, d\theta \\
&= c_1 v(y) \int_{0}^{2\pi} \int_{0}^{\pi} \sin(\phi) \, d\phi \, d\theta \\
&= 2c_1 v(y) \int_{0}^{2\pi} \, d\theta \\
&= 4\pi c_1 v(y).
\end{align*}
\]

From this we see that \( g \) is a solution of (11.83) if and only if \( c_1 = 1/(4\pi) \).
11.6 The Green’s function for the Laplacian on a bounded domain

1. Suppose \( u \) satisfies

\[
-\Delta u = f(x) \text{ in } \Omega,
\]

\[
\alpha_1 u + \alpha_2 \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega,
\]

and let \( v \) be any test function in \( C^2(\bar{\Omega}) \). Then

\[
-\int_\Omega (\Delta u)v = \int_\Omega f v \Rightarrow -\int_{\partial \Omega} \frac{\partial u}{\partial n} v + \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v.
\]

The boundary condition yields

\[
-\frac{\partial u}{\partial n} = \frac{\alpha_1}{\alpha_2} u \text{ on } \partial \Omega,
\]

and hence we obtain

\[
\frac{\alpha_1}{\alpha_2} \int_{\partial \Omega} u v + \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v,
\]

as desired.

3. Let \( G \) be the free-space Green’s function for the Laplacian and define

\[
w(x; y) = \alpha_1 G(x; y) + \alpha \frac{\partial G}{\partial n_x}(x; y)
\]

for all \( x \in \partial \Omega \) and \( y \in \Omega \). Assume that \( u(x) = v_\Omega(x; y) \) satisfies

\[
-\Delta u = 0 \text{ in } \Omega
\]

\[
\alpha_1 u + \alpha_2 \frac{\partial u}{\partial n} = w(x; y) \text{ on } \partial \Omega,
\]

where

\[
w(x; y) = \alpha_1 G(x; y) + \alpha_2 \frac{\partial G}{\partial n_x}(x; y).
\]

Define

\[
u_2(x) = \int_\Omega v_\Omega(x; y)f(y) dy.
\]

Then, for all \( v \in C^2(\bar{\Omega}) \),

\[
-\int_\Omega (\Delta_x v_\Omega) v = 0
\]

\[
\Rightarrow -\int_{\partial \Omega} \frac{\partial v_\Omega}{\partial n_x}(x; y)v(x) d\sigma_x + \int_\Omega \nabla_x v_\Omega(x; y) \cdot \nabla v(x) dx = 0
\]

\[
\Rightarrow \frac{\alpha_1}{\alpha_2} \int_{\partial \Omega} v_\Omega(x; y)v(x) d\sigma_x - \frac{1}{\alpha_2} \int_{\partial \Omega} w(x; y)v(x) d\sigma_x + \int_\Omega \nabla_x v_\Omega(x; y) \cdot \nabla v(x) dx = 0
\]

\[
\Rightarrow \frac{\alpha_1}{\alpha_2} \int_{\partial \Omega} v_\Omega(x; y)v(x) d\sigma_x + \int_\Omega \nabla_x v_\Omega(x; y) \cdot \nabla v(x) dx = \frac{1}{\alpha_2} \int_{\partial \Omega} w(x; y)v(x) d\sigma_x.
\]

Notice how we used the boundary condition satisfied by \( v_\Omega \) to substitute for \( \frac{\partial v_\Omega}{\partial n_x} \). We now multiply both sides of this last equation by \( f(y) \) and integrate over \( \Omega \) to obtain

\[
\frac{\alpha_1}{\alpha_2} \int_{\partial \Omega} \left( \int_{\partial \Omega} v_\Omega(x; y)v(x) d\sigma_x \right) f(y) dy + \int_\Omega \left( \int_\Omega \nabla_x v_\Omega(x; y) \cdot \nabla v(x) dx \right) f(y) dy
\]

\[
= \frac{1}{\alpha_2} \int_{\partial \Omega} \left( \int_{\partial \Omega} w(x; y)v(x) d\sigma_x \right) f(y) dy.
\]

Changing the order of integration in the two integrals on the left yields

\[
\frac{\alpha_1}{\alpha_2} \int_{\partial \Omega} \left( \int_{\partial \Omega} v_\Omega(x; y) f(y) dy \right) v(x) d\sigma_x + \int_\Omega \left( \int_\Omega \nabla_x v_\Omega(x; y) f(y) dy \right) \cdot \nabla v(x) dx
\]

\[
= \frac{1}{\alpha_2} \int_{\partial \Omega} \left( \int_{\partial \Omega} w(x; y)v(x) d\sigma_x \right) f(y) dy.
\]
11.6. THE GREEN’S FUNCTION FOR THE LAPLACIAN ON A BOUNDED DOMAIN

Since
\[ \int_{\Omega} \nabla_x v_\Omega(x, y) f(y) \, dy = \nabla \left( \int_{\Omega} v_\Omega(x, y) f(y) \, dy \right) = \nabla u_2(x), \]
we obtain
\[ \frac{\alpha_1}{\alpha_2} \int_{\partial\Omega} u_2(x, y) d\sigma_x + \int_{\Omega} \nabla u_2(x) \cdot \nabla v(x) \, dx \]
\[ = \frac{1}{\alpha_2} \int_{\Omega} \left( \int_{\partial\Omega} w(x, y) v(x) \, d\sigma_x \right) f(y) \, dy, \]
as desired.

5. Let \( G \) be the free-space Green’s function for the Laplacian, and suppose \( v_\Omega(x, y) \) satisfies the BVP
\[ -\Delta_x v_\Omega = 0 \text{ in } \Omega, \]
\[ v_\Omega(x, y) = G(x; y) \text{ on } \partial\Omega \]
for all \( y \in \Omega \). (Notice that \( G(x; y) \) is a continuous function of \( x \in \partial\Omega \) for all \( y \in \Omega \), and therefore \( v_\Omega \) is well-defined.) We define \( G_\Omega(x; y) = G(x; y) - v_\Omega(x; y) \), and we will show that \( G_\Omega \) is the Green’s function for the BVP
\[ -\Delta u = f(x) \text{ in } \Omega, \]
\[ u = 0 \text{ on } \partial\Omega. \]

The weak form of the last BVP is
\[ u \in C^2_{\Omega} (\overline{\Omega}), \quad \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx = \int_{\Omega} f(x) v(x) \, dx \text{ for all } v \in C^2_{\Omega} (\overline{\Omega}). \]
We prove that \( G_\Omega \) is the desired Green’s function by proving that
\[ u(x) = \int_{\Omega} G_\Omega(x; y) f(y) \, dy \]
satisfies the above variational problem. We have
\[ u(x) = \int_{\Omega} G(x; y) f(y) \, dy - \int_{\Omega} v_\Omega(x; y) f(y) \, dy = u_1(x) - u_2(x), \]
where
\[ u_1(x) = \int_{\Omega} G(x; y) f(y) \, dy, \quad u_2(x) = \int_{\Omega} v_\Omega(x; y) f(y) \, dy. \]
Letting \( V \) be the space of all smooth functions with compact support, we have
\[ \int_{\mathbb{R}^2} \nabla u_1(x) \cdot \nabla v(x) \, dx = \int_{\mathbb{R}^2} f(x) v(x) \, dx \text{ for all } v \in V, \]
where \( f \) has been extended to be zero outside of \( \Omega \). This last variational equation holds because \( G \) is the free-space Green’s function. For any \( v \in C^2_{\Omega} (\overline{\Omega}) \), we can extend \( v \) to be identically zero outside of \( \overline{\Omega} \) and thereby obtain an element \( v \) of \( V \). We then have
\[ \int_{\Omega} \nabla u_1(x) \cdot \nabla v(x) \, dx = \int_{\Omega} f(x) v(x) \, dx \text{ for all } v \in C^2_{\Omega} (\overline{\Omega}). \]
On the other hand, we have
\[ -\Delta u_2(x) = 0 \text{ for all } x \in \Omega; \]
multiplying by \( v \in C^2_{\Omega} (\overline{\Omega}) \) and integrating yields
\[ -\int_{\Omega} \Delta u_2(x) v(x) \, dx = 0 \text{ for all } v \in C^2_{\Omega} (\overline{\Omega}). \]
Integrating by parts on the left, we obtain
\[ \int_{\Omega} \nabla u_2(x) \cdot \nabla v(x) \, dx = 0 \text{ for all } v \in C^2_{\Omega} (\overline{\Omega}). \]
CHAPTER 11. PROBLEMS IN MULTIPLE SPATIAL DIMENSIONS

(Notice that the boundary term vanishes because \( v = 0 \) on \( \partial \Omega \).) Putting together the variational equations satisfies by \( u_1 \) and \( u_2 \), we see that \( u = u_1 - u_2 \) satisfies

\[
\int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx = \int_{\Omega} f(x)v(x) \, dx \quad \text{for all } v \in C^2_0(\Omega).
\]

Also, for \( x \in \partial \Omega \), we have

\[
u(x) = \int_{\Omega} G(x; y)f(y) \, dy - \int_{\partial \Omega} v_{\Omega}(x; y)f(y) \, dy = \int_{\Omega} G(x; y)f(y) \, dy - \int_{\partial \Omega} G(x; y)f(y) \, dy = 0.
\]

This shows that \( u \in C^2_0(\Omega) \); hence \( u \) satisfies the given BVP, which shows that \( G_{\Omega} \) is the desired Green’s function.

7. Green’s second identity is

\[
\int_{\Omega} (v \Delta u - u \Delta v) = \int_{\partial \Omega} \left( v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right).
\]

Let \( G_{\Omega} \) be the Green’s function for the BVP

\[
-\Delta u = f(x) \text{ in } \Omega \quad \alpha_1 u + \alpha_2 \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega,
\]

and, for \( y, z \in \Omega \), define \( u(x) = G_{\Omega}(x; y) \), \( v(x) = G_{\Omega}(x; z) \). Since \( G_{\Omega}(x; y) \) satisfies

\[
-\Delta_x G_{\Omega} = \delta(x - y) \text{ in } \Omega \quad \alpha_1 G_{\Omega} + \alpha_2 \frac{\partial G_{\Omega}}{\partial n_x} = 0, \text{ on } \partial \Omega,
\]

we see that

\[
\int_{\Omega} v \Delta u = -v(y) = -G(y; z),
\]

\[
\int_{\Omega} u \Delta v = -u(z) = -G(z; y).
\]

Therefore,

\[
\int_{\Omega} (v \Delta u - u \Delta v) = G(z; y) - G(y; z).
\]

We also have

\[
\frac{\partial u}{\partial n} = -\frac{\alpha_1}{\alpha_2} u \text{ on } \partial \Omega,
\]

or

\[
\frac{\partial G_{\Omega}}{\partial n_x}(x; y) = -\frac{\alpha_1}{\alpha_2} G_{\Omega}(x; y).
\]

Similarly,

\[
\frac{\partial G_{\Omega}}{\partial n_x}(x; z) = -\frac{\alpha_1}{\alpha_2} G_{\Omega}(x; z).
\]

Therefore,

\[
\int_{\partial \Omega} \left( v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) = \int_{\partial \Omega} \left( -\frac{\alpha_1}{\alpha_2} G_{\Omega}(x; z) G_{\Omega}(x; y) + \frac{\alpha_1}{\alpha_2} G_{\Omega}(x; y) G_{\Omega}(x; z) \right) \, d\sigma_x = 0.
\]

Thus, by Green’s second identity,

\[
G(z; y) - G(y; z) = 0.
\]

that is, \( G(z; y) = G(y; z) \), as desired.

9. Let \( \Omega \subset \mathbb{R}^2 \) be given, and let \( u \) be the solution of

\[
-\Delta u = 0 \text{ in } \Omega, \quad u = \phi(x) \text{ on } \partial \Omega.
\]
11.6. THE GREEN’S FUNCTION FOR THE LAPLACIAN ON A BOUNDED DOMAIN

Let \( v(x) = G_\Omega(x;y) \), where \( G_\Omega \) is the Green’s function for the BVP

\[
-\Delta u = f(x) \text{ in } \Omega, \\
u = 0 \text{ on } \partial\Omega.
\]

Finally, define \( \Omega_\epsilon = \{ x \in \Omega : \| x - y \| > \epsilon \} \), where \( y \in \Omega \) is given and \( \epsilon > 0 \) is small enough that \( B_\epsilon(y) \subset \Omega \). Notice that \( \Delta u = 0 \) in \( \Omega \), \( \Delta v = 0 \) in \( \Omega_\epsilon \), and \( v(x) = 0 \) for all \( x \in \partial\Omega \). Applying Green’s second identity to \( u \) and \( v \) yields

\[
\int_{\Omega} (v\Delta u - u\Delta v) = \int_{\partial\Omega} \left( v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) d\sigma_x
\]

\[
\Rightarrow 0 = \int_{\partial\Omega} \left( v(x)\frac{\partial u}{\partial n}(x) - u(x)\frac{\partial v}{\partial n}(x) \right) d\sigma_x + \int_{S_\epsilon(y)} \left( v(x)\frac{\partial u}{\partial n}(x) - u(x)\frac{\partial v}{\partial n}(x) \right) d\sigma_x
\]

\[
\Rightarrow 0 = \int_{\partial\Omega} \left( 0 - \phi(x)\frac{\partial G_\Omega}{\partial n_x}(x;y) \right) d\sigma_x + \int_{S_\epsilon(y)} \left( G_\Omega(x;y)\frac{\partial u}{\partial n}(x) - u(x)\frac{\partial G_\Omega}{\partial n_x}(x;y) \right) d\sigma_x
\]

\[
\Rightarrow \int_{\partial\Omega} \phi(x)\frac{\partial G_\Omega}{\partial n_x}(x;y) d\sigma_x = \int_{S_\epsilon(y)} G_\Omega(x;y)\frac{\partial u}{\partial n}(x) d\sigma_x - \int_{S_\epsilon(y)} u(x)\frac{\partial G_\Omega}{\partial n_x}(x;y) d\sigma_x.
\]

Now, recall that \( G_\Omega = G - v_\Omega \), where \( G \) is the free-space Green’s function for the Laplacian and \( v_\Omega(x;y) \) satisfies

\[
-\Delta v_\Omega = 0 \text{ in } \Omega, \\
v_\Omega(x;y) = G(x;y) \text{ on } \partial\Omega.
\]

We therefore have

\[
\int_{\partial\Omega} \phi(x)\frac{\partial G_\Omega}{\partial n_x}(x;y) d\sigma_x = \int_{S_\epsilon(y)} G(x;y)\frac{\partial u}{\partial n}(x) d\sigma_x - \int_{S_\epsilon(y)} u(x)\frac{\partial G_\Omega}{\partial n_x}(x;y) d\sigma_x
\]

The integrands for the last two integrals are continuous, with no singularities as \( \epsilon \to 0^+ \). Therefore, these two integrals tend to zero as \( \epsilon \to 0^+ \). We have

\[
G(x;y) = -\frac{1}{2\pi} \ln (\| x - y \|),
\]

and therefore \( G(x;y) \) is constant for \( x \in S_\epsilon(y) \). It follows that

\[
\int_{S_\epsilon(y)} G(x;y)\frac{\partial u}{\partial n}(x) d\sigma_x
\]

is just a multiple of

\[
\int_{S_\epsilon(y)} \frac{\partial u}{\partial n}(x) d\sigma_x = \int_{B_\epsilon(y)} \Delta u(x) dx = 0
\]

(applying the divergence theorem and using the fact that \( \Delta u = 0 \) in \( \Omega \)). It remains only to calculate

\[
\int_{S_\epsilon(y)} u(x)\frac{\partial G_\Omega}{\partial n_x}(x;y) d\sigma_x.
\]

Now,

\[
\nabla_x G(x;y) = -\frac{x - y}{2\pi\| x - y \|^2}
\]

and, on \( S_\epsilon(y) \),

\[
\n_x = -\frac{x - y}{\| x - y \|}
\]

(notice that this is the outward-pointing normal to \( \Omega \)). Therefore,

\[
\frac{\partial G_\Omega}{\partial n_x}(x;y) = \nabla_x G(x;y) \cdot n_x = \frac{1}{2\pi\epsilon}.
\]
13. In Chapter 9, it was shown that the solution to

\[ \Delta u = 0 \] in \( B_R(0) \),

\[ u = \psi(x) \] on \( S_R(0) \),

then

\[ u(x) = \int_{\partial B_R(0)} \frac{R^2 - ||x||^2}{2\pi R||x - y||^2} \phi(y) \, d\sigma_y. \]

Now suppose that \( u \) is represented as a function of polar coordinates \((r, \theta)\). Let the circle \( S_R(0) \) be represented by \( y = (R \cos(\psi), R \sin(\psi)) \), \( 0 \leq \psi \leq 2\pi \). Then

\[ d\sigma_y = \sqrt{\left( \frac{\partial y_1}{\partial \psi} \right)^2 + \left( \frac{\partial y_2}{\partial \psi} \right)^2} \, d\psi = \sqrt{R^2 \cos^2(\psi) + R^2 \sin^2(\psi)} \, d\psi = Rd\psi. \]

With \( x \) represented by the polar coordinates \((r, \theta)\), we have \( ||x||^2 = r^2 \) and

\[ ||x - y||^2 = (r \cos(\theta) - R \cos(\psi))^2 + (r \sin(\theta) - R \sin(\psi))^2 = r^2 + R^2 - 2rR \cos(\psi - \theta) \]

(where we used the trigonometric identity \( \cos(\theta) \cos(\psi) + \sin(\theta) \sin(\psi) = \cos(\psi - \theta) \)). Changing variables from \( y \) to \((r, \theta)\) thus yields

\[ \int_{\partial B_R(0)} \frac{R^2 - ||y||^2}{2\pi R||y - x||^2} \phi(y) \, d\sigma_y = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2rR \cos(\psi - \theta)} \phi(\psi) \, d\psi, \]

as desired.

13. In Chapter 9, it was shown that the solution to

\[ -\frac{d}{dx} \left( P(x) \frac{du}{dx} \right) + R(x)u = 0, \quad a < x < b, \]

\[ u(a) = u_a, \]

\[ u(b) = u_b \]

is

\[ u(y) = P(a)u_a \frac{\partial G}{\partial y}(y; a) - P(b)u_b \frac{\partial G}{\partial y}(y; b), \]

where \( G \) is the Green’s function for the inhomogeneous differential equation with homogeneous boundary conditions. If we take \( P(x) = 1 \), we obtain

\[ u(y) = -\left( \frac{\partial G}{\partial y}(y; a)u_a + \frac{\partial G}{\partial y}(y; b)u_b \right). \]

Thus \( u \) is obtained by “integrating” (that is, summing) the normal derivative of the Green’s function, times the boundary “function” over the boundary of \( \Omega = (a, b) \) (that is, the set \( \{a, b\} \)). Notice that the unit normal to \( \Omega \) at \( x = b \) is \( n_1 = 1 \), while at \( x = a \) it is \( n = -1 \), so that

\[ -\frac{\partial G}{\partial y}(y; a). \]
11.7. GREEN’S FUNCTION FOR THE WAVE EQUATION

is the normal derivative of $G$ at $x = a$, while

$$\frac{\partial G}{\partial y}(y; b)$$

is the normal derivative of $G$ at $x = b$. Thus formula (9.24) is exactly analogous to formula (11.103).

15. If $\Omega = B_R(z)$ for some $z \in \mathbb{R}^2$, then formula (11.107) can be modified to show that the solution of

$$-\Delta u = f(x), \ x \in B_R(z),$$

$$u(x) = 0, \ x \in S_R(z)$$

is

$$u(x) = \int_{S_R(z)} \frac{R^2 - \|x - y\|^2}{2\pi R \|x - y\|^2} \phi(y) \, d\sigma_y.$$

Substituting $x = z$, we see that $\|z - y\|^2 = R^2$ (since $y \in S_R(z)$, and of course $\|z - z\|^2 = 0$; therefore,

$$u(z) = \int_{S_R(z)} \frac{R^2}{2\pi R} \phi(y) \, d\sigma_y = \frac{1}{2\pi} \int_{S_R(z)} \phi(y) \, d\sigma_y.$$

Thus $u(z)$ is the average value of $u$ over the circle $S_R(z)$ (recall that $u(y) = \phi(y)$ on $S_R(z)$).

11.7 Green’s function for the wave equation

1. Let $\psi : \mathbb{R}^3 \to \mathbb{R}$, $c \in \mathbb{R}$, and $y \in \mathbb{R}^3$ be given. Then, by the chain rule,

$$\frac{\partial}{\partial t} (\psi(x + ct)) = \sum_{i=1}^3 \frac{\partial \psi}{\partial x_i}(x + ct) \frac{\partial}{\partial t}(x_i + cy_i)$$

$$= \sum_{i=1}^3 \frac{\partial \psi}{\partial x_i}(x + ct)(cy_i)$$

$$= c \sum_{i=1}^3 \frac{\partial \psi}{\partial x_i}(x + ct)y_i = \nabla \psi(x + ct) \cdot y.$$

Similarly,

$$\frac{\partial^2}{\partial t^2} (\psi(x + ct)) = \frac{\partial}{\partial t} \left( c \sum_{i=1}^3 \frac{\partial \psi}{\partial x_i}(x + ct)y_i \right)$$

$$= c \sum_{i=1}^3 \frac{\partial}{\partial t} \left( \frac{\partial \psi}{\partial x_i}(x + ct)y_i \right)$$

$$= c \sum_{i=1}^3 \sum_{j=1}^3 \frac{\partial^2 \psi}{\partial x_j \partial x_i}(x + ct)y_i \frac{\partial}{\partial t}(x_j + cy_j)$$

$$= c \sum_{i=1}^3 \sum_{j=1}^3 \frac{\partial^2 \psi}{\partial x_j \partial x_i}(x + ct)y_i cy_j$$

$$= c^2 \sum_{i=1}^3 y_i \left( \sum_{j=1}^3 \frac{\partial^2 \psi}{\partial x_j \partial x_i}(x + ct)y_j \right)$$

$$= c^2 \sum_{i=1}^3 y_i (\nabla^2 \psi(x + ct)y_i)$$

$$= c^2 y \cdot \nabla^2 \psi(x + ct)y.$$

3. We wish to derive the solution of

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0, \ x \in \mathbb{R}^2, \ t > 0,$$

$$u(x, 0) = 0, \ x \in \mathbb{R}^2,$$

$$\frac{\partial u}{\partial t}(x, 0) = \psi(x), \ x \in \mathbb{R}^2.$$
CHAPTER 11. PROBLEMS IN MULTIPLE SPATIAL DIMENSIONS

We can follow the same reasoning as in 11.7.2. The solution of the same IVP in three dimensions is

\[ u(x, t) = \frac{t}{4\pi} \int_{\partial B_1(0)} \psi(x + cty) \, d\sigma_y. \]

If, in this formula, \( \psi \) does not depend on \( x_3 \), then neither does \( u \), and it is easy to see that \( u \), regarded as a function on \( \mathbb{R}^2 \), is the solution of the IVP given above. Therefore, it remains only to write \( u \) as an integral over a two-dimensional region, namely, \( B_1(0) \). If \( S \) represents the upper hemisphere of \( \partial B_1(0) \), then

\[
\int_{\partial B_1(0)} \psi(x + cty) \, d\sigma_y = 2 \int_S \psi(x + cty) \, d\sigma_y
\]

\[
= 2 \int_{-1}^{1} \frac{\psi(x_1 + cty_1, x_2 + cty_2)}{\sqrt{1 - y_1^2 - y_2^2}} \, dy_2 \, dy_1
\]

\[
= 2 \int_{B_1(0)} \frac{\psi(x + cty)}{\sqrt{1 - \|y\|^2}} \, dy.
\]

It follows that

\[ u(x, t) = \frac{t}{2\pi} \int_{B_1(0)} \frac{\psi(x + cty)}{\sqrt{1 - \|y\|^2}} \, dy. \]

Performing the change of variables \( z = x + cty \) yields the alternate formula

\[ u(x, t) = \frac{1}{2\pi c} \int_{B_{ct}(x)} \frac{\psi(z)}{\sqrt{c^2 t^2 - \|z - x\|^2}} \, dy. \]

5. The solution of

\[
\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0, \quad x \in \mathbb{R}^3, \quad t > 0,
\]

\[ u(x, 0) = 0, \quad x \in \mathbb{R}^3,
\]

\[ \frac{\partial u}{\partial t}(x, 0) = \psi(x), \quad x \in \mathbb{R}^3.
\]
is

\[ u(x, t) = \frac{t}{4\pi(c t)^2} \int_{\partial B_{ct}(x)} \psi(z) \, d\sigma z. \]

Let

\[ \psi(x) = \begin{cases} 1, & \|x\| < 1, \\ 0, & \text{otherwise}, \end{cases} \]

and let \( u \) be the corresponding solution of the above IVP. Suppose first that \( x \in \mathbb{R}^3, \|x\| > R \), and \( t < (\|y\| - R)/c \). This last condition implies that \( ct < \|x\| - R \), and therefore, if \( z \in S_{ct}(x) \), then

\[ \|z - x\| = ct < \|x\| - R. \]

It is easy to see that this condition implies that \( \|z\| > R \) (since \( z \) is closer to \( x \) than \( x \) is to \( S_R(0) \)). Therefore,

\[ z \in S_{ct}(x) \Rightarrow \psi(z) = 0, \]

and it follows from the formula for \( u(x, t) \) that \( u(x, t) = 0 \).

Now suppose \( t > (\|x\| + R)/c \), that is, \( ct > \|x\| + R \). Then

\[ z \in S_{ct}(x) \Rightarrow \|z - x\| = ct > \|x\| + R, \]

which implies that \( \|z\| > R \) (since \( \|z - x\| \leq \|z\| + \|x\| \)). Therefore, in this case also,

\[ z \in S_{ct}(x) \Rightarrow \psi(z) = 0, \]

and we see that \( u(x, t) = 0 \).

7. The causal Green’s function is

\[
G(x, t; y, s) = \begin{cases} \sum_{n=1}^{\infty} \frac{\sin \left( c\sqrt{x_n^2 (t-s)} \right)}{c\sqrt{x_n}} \psi_n(x) \psi_n(y), & t > s, \\ 0, & t < s. \end{cases}
\]
11.8 Green’s functions for the heat equation

1. We wish to show that the Gaussian kernel,
\[ S(x, t) = (4k\pi t)^{-n/2}e^{-\|x\|^2/(4kt)} \]
satisfies the homogeneous heat equation in \( \mathbb{R}^n \). We have
\[
\frac{\partial S}{\partial t}(x, t) = \frac{1}{(4k\pi t)^{-n/2}e^{-\|x\|^2/(4kt)}} \left( \frac{\|x\|^2}{4kt^2} - \frac{n}{2t} \right),
\]
\[
\frac{\partial S}{\partial x_i}(x, t) = \frac{1}{(4k\pi t)^{-n/2}e^{-\|x\|^2/(4kt)}} \left( x_i - \frac{1}{2kt} \right),
\]
\[
\frac{\partial^2 S}{\partial x_i \partial x_j}(x, t) = \frac{1}{(4k\pi t)^{-n/2}e^{-\|x\|^2/(4kt)}} \left( x_i x_j - \frac{1}{2kt} - \frac{1}{2kt^2} \right),
\]
\[
\Delta S(x, t) = \frac{1}{(4k\pi t)^{-n/2}e^{-\|x\|^2/(4kt)}} \left( \frac{n}{2kt} \right).
\]
From these formulas, it follows immediately that
\[
\frac{\partial S}{\partial t}(x, t) - k\Delta S(x, t) = 0.
\]

3. We have seen that the IBVP
\[
\frac{\partial v}{\partial t} - k\Delta v = f(x, T-t), \quad x \in \Omega, \quad 0 < t < T,
\]
\[
v(x, 0) = \phi(x), \quad x \in \Omega,
\]
\[
v(x, t) = 0, \quad x \in \partial \Omega, \quad 0 < t < T
\]
has a unique solution \( v \). Let us define \( u(x, t) = v(x, T-t) \). Then
\[
\frac{\partial u}{\partial t}(x, t) = -\frac{\partial v}{\partial t}(x, T-t),
\]
\[
\Delta u(x, t) = \Delta v(x, T-t),
\]
and hence
\[
-\frac{\partial u}{\partial t}(x, t) - k\Delta u(x, t) = \frac{\partial v}{\partial t}(x, T-t) - k\Delta v(x, T-t) = f(x, T-(T-t)) = f(x, t).
\]
Also, \( u(x, T) = v(x, 0) = \phi(x) \) for all \( x \in \Omega \), and \( u(x, t) = v(x, T-t) = 0 \) for all \( x \in \partial \Omega \) and \( 0 < t < T \). Thus \( u \) satisfies
\[
-\frac{\partial u}{\partial t} - k\Delta u = f(x, t), \quad x \in \Omega, \quad 0 < t < T,
\]
\[
u(x, T) = \phi(x), \quad x \in \Omega,
\]
\[
u(x, t) = 0, \quad x \in \partial \Omega, \quad 0 < t < T,
\]
which shows that this IBVP has a solution.

5. Let \( u, v \) be smooth functions defined for \( x \in \overline{\Omega} \) and \( 0 \leq t \leq T \). We have
\[
\int_0^T \int_\Omega \left\{ u \left( \frac{\partial v}{\partial t} - k\Delta v \right) - v \left( \frac{\partial u}{\partial t} - k\Delta u \right) \right\} \, dx \, dt
\]
\[
= \int \int_\Omega u \frac{\partial v}{\partial t} \, dx \, dt - k \int_\Omega u \Delta v \, dx \, dt + k \int_\Omega v \Delta u \, dx \, dt + \int_\Omega u \frac{\partial u}{\partial t} \, dx \, dt + k \int_\Omega v \frac{\partial u}{\partial t} \, dx \, dt
\]
\[
= \int_\Omega \left( u \int_0^T \frac{\partial v}{\partial t} \, dt \right) \, dx - k \int_\Omega \int_0^T \frac{\partial u}{\partial t} \, dt \, dx + k \int_0^T \int_\Omega \frac{\partial u}{\partial t} \, dx \, dt
\]
\[
= \int_\Omega \int_0^T \left( u \left( \nabla u \cdot \nabla v \right) - \nabla u \cdot \nabla v \, dx \right) \, dt + k \int_\Omega \int_0^T \frac{\partial u}{\partial t} \, dx \, dt
\]
\[
= \int_\Omega \int_0^T \left( v \left( \nabla u \cdot \nabla v \right) - \nabla u \cdot \nabla v \, dx \right) \, dt + k \int_\Omega \int_0^T \frac{\partial U}{\partial t} \, dx \, dt
\]
\[
= \int_\Omega (u(x, T)v(x, T) - u(x, 0)v(x, 0)) \, dx + k \int_\Omega \int_\partial \Omega \left( \frac{\partial u}{\partial n} - \frac{\partial v}{\partial n} \right) \, d\sigma_x.
\]
Chapter 12

More about Fourier Series

12.1 The complex Fourier series

1. The complex Fourier series of $f$ is

$$
\sum_{n=-\infty}^{\infty} c_n e^{i\pi nx},
$$

where $c_0 = \frac{2}{3}$ and

$$
c_n = -\frac{2(-1)^n}{n^2 \pi^2}, \quad n = \pm 1, \pm 2, \ldots.
$$

The errors in approximating $f$ by a partial Fourier series are shown in Figure 12.1.

![Figure 12.1](image)

Figure 12.1: The error in approximating $f(x) = 1 - x^2$ by its partial complex Fourier series with $N = 10$ (top), $N = 20$ (middle), and $N = 40$ (bottom). (See Exercise 12.1.1.)

3. The complex Fourier coefficients of $f$ are

$$
c_n = \frac{(-1)^{n+1} \sin(1)}{n\pi - 1}.
$$

The magnitudes of the error in approximating $f$ by a partial Fourier series are shown in Figure 12.2.

5. Let

$$
g(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \left( \frac{n\pi x}{\ell} \right) + b_n \sin \left( \frac{n\pi x}{\ell} \right) \right),
$$
where
\[
\begin{align*}
    a_0 &= \frac{1}{2L} \int_{-L}^{L} g(x) \, dx, \\
    a_n &= \frac{1}{L} \int_{-L}^{L} g(x) \cos \left( \frac{n\pi x}{L} \right) \, dx, \quad n = 1, 2, 3, \ldots, \\
    b_n &= \frac{1}{L} \int_{-L}^{L} g(x) \sin \left( \frac{n\pi x}{L} \right) \, dx, \quad n = 1, 2, 3, \ldots,
\end{align*}
\]
(see (6.25) in the text), and similarly let
\[
h(x) = p_0 + \sum_{n=1}^{\infty} \left( p_n \cos \left( \frac{n\pi x}{L} \right) + q_n \sin \left( \frac{n\pi x}{L} \right) \right).
\]

The complex Fourier coefficient of \(f\) is given by
\[
c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i\pi nx/L} \, dx.
\]
For \(n > 0\), we have
\[
c_n = \frac{1}{2L} \int_{-L}^{L} \left( g(x) + ih(x) \right) \left( \cos \left( \frac{n\pi x}{L} \right) - i \sin \left( \frac{n\pi x}{L} \right) \right) \, dx
\]
\[
= \frac{1}{2} \left( \frac{1}{L} \int_{-L}^{L} g(x) \cos \left( \frac{n\pi x}{L} \right) \, dx - i \frac{1}{L} \int_{-L}^{L} g(x) \sin \left( \frac{n\pi x}{L} \right) \, dx + i \frac{1}{L} \int_{-L}^{L} h(x) \cos \left( \frac{n\pi x}{L} \right) \, dx + \frac{1}{L} \int_{-L}^{L} h(x) \sin \left( \frac{n\pi x}{L} \right) \, dx \right)
\]
\[
= \frac{1}{2} \left( a_n + q_n + i(p_n - b_n) \right).
\]
Similarly, if \(n > 0\), then
\[
c_{-n} = \frac{1}{2} \left( a_n - q_n + i(p_n + b_n) \right).
\]
Finally,
\[
c_0 = a_0 + ip_0.
\]

7. We have
\[
\sum_{n=0}^{N-1} \left( e^{\pi i n/N} \right)^2 = \sum_{n=0}^{N-1} \left( e^{2\pi i n/N} \right)^n
\]
\[
= \frac{e^{2\pi i} - 1}{e^{2\pi i} - 1}.
\]
using the formula for the sum of a finite geometric series:

\[
\sum_{n=0}^{m} r^n = \frac{r^{m+1} - 1}{r - 1}.
\]

Moreover, since \(e^{i\theta}\) is a 2\(\pi\)-periodic function of \(\theta\), we see that \(e^{2\pi i} = 1\) and hence that

\[
\sum_{n=0}^{N-1} \left( e^{\frac{i\pi jn}{N}} \right)^2 = 0.
\]

On the other hand, by Euler’s formula,

\[
\sum_{n=0}^{N-1} \left( e^{\frac{i\pi jn}{N}} \right)^2 = \sum_{n=0}^{N-1} \left( \cos \left( \frac{\pi jn}{N} \right) + i \sin \left( \frac{\pi jn}{N} \right) \right)^2
\]

\[
= \sum_{n=0}^{N-1} \left( \cos^2 \left( \frac{\pi jn}{N} \right) - \sin^2 \left( \frac{\pi jn}{N} \right) + 2i \cos \left( \frac{\pi jn}{N} \right) \sin \left( \frac{\pi jn}{N} \right) \right)
\]

\[
= \sum_{n=0}^{N-1} \cos^2 \left( \frac{\pi jn}{N} \right) - \sum_{n=0}^{N-1} \sin^2 \left( \frac{\pi jn}{N} \right) + 2i \sum_{n=0}^{N-1} \cos \left( \frac{\pi jn}{N} \right) \sin \left( \frac{\pi jn}{N} \right).
\]

Since this expression equals zero, we must have

\[
\sum_{n=0}^{N-1} \cos^2 \left( \frac{\pi jn}{N} \right) - \sum_{n=0}^{N-1} \sin^2 \left( \frac{\pi jn}{N} \right) = 0,
\]

\[
\sum_{n=0}^{N-1} \cos \left( \frac{\pi jn}{N} \right) \sin \left( \frac{\pi jn}{N} \right) = 0.
\]

The second equation is one of the results we set out to prove. The first equation can be written, using the trigonometric identity \(\sin^2 (\theta) = 1 - \cos^2 (\theta)\), as

\[
\sum_{n=0}^{N-1} \cos^2 \left( \frac{\pi jn}{N} \right) - \sum_{n=0}^{N-1} \left( 1 - \cos^2 \left( \frac{\pi jn}{N} \right) \right) = 0,
\]

which simplifies to

\[
2 \sum_{n=0}^{N-1} \cos^2 \left( \frac{\pi jn}{N} \right) = N,
\]

or

\[
\sum_{n=0}^{N-1} \cos^2 \left( \frac{\pi jn}{N} \right) = \frac{N}{2},
\]

which is the desired result. The result

\[
\sum_{n=0}^{N-1} \sin^2 \left( \frac{\pi jn}{N} \right) = \frac{N}{2}
\]

then follows.

### 12.2 Fourier series and the FFT

1. The graph of \(|F_n|\) is shown in Figure 12.3.

3. As shown in the text,

\[c_n \doteq F_n, \quad n = -16, -15, \ldots, 15,\]
CHAPTER 12. MORE ABOUT FOURIER SERIES

Figure 12.3: The magnitude of the sequence \( \{F_n\} \) from Exercise 12.2.1.

Figure 12.4: The function \( f(x) = x(1 - x^2) \) (the curve) and the estimates of \( f(x_j) \) (the circles) computed from the complex Fourier coefficients of \( f \) (see Exercise 12.2.3).

where \( \{F_n\}_{n=-16}^{15} \) is the DFT of \( \{f_j\}_{j=-16}^{15} \).

\[
f_j = f(x_j), \quad x_j = -1 + \frac{j}{16}, \quad j = -16, -15, \ldots, 15
\]

(note that \( f(-1) = f(1) \)). Therefore, we can estimate \( \{f(x_j)\} \) by taking the inverse DFT (using the inverse FFT) of \( \{c_n\} \). The results are shown in Figure 12.4.

5. We have

\[
\sum_{n=0}^{N-1} A_n e^{2\pi ijn/N} = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} a_m e^{-2\pi imn/N} e^{2\pi ijn/N} = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} a_m e^{2\pi i(j-m)n/N} = \frac{1}{N} \sum_{m=0}^{N-1} a_m \left( \sum_{n=0}^{N-1} e^{2\pi i(j-m)n/N} \right).
\]

If \( j = m \), then

\[
\sum_{n=0}^{N-1} e^{2\pi (j-m)n/N} = \sum_{n=0}^{N-1} 1 = N,
\]
while if \( j \neq m \), then this sum is a finite geometric series:

\[
\sum_{n=0}^{N-1} e^{2\pi i(j-m)n/N} = \sum_{n=0}^{N-1} (e^{2\pi i(j-m)/N})^n = \frac{(e^{2\pi i(j-m)/N})^N - 1}{e^{2\pi i(j-m)/N} - 1} = \frac{e^{2\pi i(j-m)} - 1}{e^{2\pi i(j-m)/N} - 1} = 0.
\]

Therefore,

\[
a_m \sum_{n=0}^{N-1} e^{2\pi i(j-m)n/N} = \begin{cases} \frac{N}{2}, & m = j, \\ 0, & m \neq j, \end{cases}
\]

and so we obtain

\[
\sum_{n=0}^{N-1} A_n e^{2\pi i n/N} = \frac{1}{N} \sum_{m=0}^{N-1} a_m \left\{ \sum_{n=0}^{N-1} e^{2\pi i(j-m)n/N} \right\} = \frac{1}{N} Na_j = a_j,
\]

as desired.

7. Below we show the exact Fourier sine coefficients of \( f \), the coefficients estimated by the DST, and the relative error. Since both \( a_n \) and the estimated \( a_n \) are zero for \( n \) even, we show only \( a_n \) for \( n \) odd.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( a_n )</th>
<th>estimated ( a_n )</th>
<th>relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.5801 \cdot 10^{-1}</td>
<td>2.5801 \cdot 10^{-1}</td>
<td>6.2121 \cdot 10^{-6}</td>
</tr>
<tr>
<td>3</td>
<td>9.5560 \cdot 10^{-3}</td>
<td>9.5511 \cdot 10^{-3}</td>
<td>5.1571 \cdot 10^{-4}</td>
</tr>
<tr>
<td>5</td>
<td>2.0641 \cdot 10^{-3}</td>
<td>2.0555 \cdot 10^{-3}</td>
<td>4.1824 \cdot 10^{-3}</td>
</tr>
<tr>
<td>7</td>
<td>7.5222 \cdot 10^{-4}</td>
<td>7.3918 \cdot 10^{-4}</td>
<td>1.7340 \cdot 10^{-2}</td>
</tr>
<tr>
<td>9</td>
<td>3.5393 \cdot 10^{-4}</td>
<td>3.3531 \cdot 10^{-4}</td>
<td>5.2608 \cdot 10^{-2}</td>
</tr>
<tr>
<td>11</td>
<td>1.9385 \cdot 10^{-4}</td>
<td>1.6778 \cdot 10^{-4}</td>
<td>1.3448 \cdot 10^{-1}</td>
</tr>
<tr>
<td>13</td>
<td>1.1744 \cdot 10^{-4}</td>
<td>8.0874 \cdot 10^{-5}</td>
<td>3.1135 \cdot 10^{-1}</td>
</tr>
<tr>
<td>15</td>
<td>7.6448 \cdot 10^{-5}</td>
<td>2.4279 \cdot 10^{-5}</td>
<td>6.8241 \cdot 10^{-1}</td>
</tr>
</tbody>
</table>

9. We have

\[
2 \sum_{n=1}^{N-1} F_n \sin \left( \frac{jn\pi}{N} \right) = \sum_{n=1}^{N-1} \sum_{m=1}^{N-1} f_m \sin \left( \frac{mn\pi}{N} \right) \sin \left( \frac{jn\pi}{N} \right) = 4 \sum_{m=1}^{N-1} f_m \sum_{n=1}^{N-1} \sin \left( \frac{mn\pi}{N} \right) \sin \left( \frac{jn\pi}{N} \right).
\]

By the hint,

\[
\sum_{n=1}^{N-1} \sin \left( \frac{mn\pi}{N} \right) \sin \left( \frac{jn\pi}{N} \right)
\]

is either \( N/2 \) or 0, depending on whether \( m = j \) or not. It then follows immediately that

\[
2 \sum_{n=1}^{N-1} F_n \sin \left( \frac{jn\pi}{N} \right) = 4 \frac{N}{2} f_j = 2N f_j,
\]

which is what we wanted to prove.
12.3 Relationship of sine and cosine series to the full Fourier series

1. The partial Fourier sine series, with 50 terms, is shown in Figure 12.5. It appears that the series converges to the function $F$ satisfying

$$F(x) = x - 2k, \ -2k - 1 < x < 2k + 1, \ k = 0, \pm 1, \pm 2, \ldots.$$ 

The period of this function is 2.

3. (a) The odd extension $f_{\text{odd}}$ is continuous on $[-\ell, \ell]$ only if $f(0) = 0$. Otherwise, $f_{\text{odd}}$ has a jump discontinuity at $x = 0$.

(b) The even extension $f_{\text{even}}$ is continuous on $[-\ell, \ell]$ for every continuous $f : [0, \ell] \to \mathbb{R}$.

5. The quarter-wave sine series of $f : [0, \ell] \to \mathbb{R}$ is the full Fourier series of the function $\tilde{f} : [-2\ell, 2\ell] \to \mathbb{R}$ obtained by first reflecting the graph of $f$ across the line $x = \ell$ (to obtain a function defined on $[0, 2\ell]$) and then taking the odd extension of the result. That is, $	ilde{f}$ is defined by

$$\tilde{f}(x) = \begin{cases} 
  f(x), & 0 \leq x \leq \ell, \\
  f(2\ell - x), & \ell < x \leq 2\ell, \\
  -f(-x), & -\ell \leq x < 0, \\
  -f(x + 2\ell), & -2\ell \leq x < -\ell.
\end{cases}$$

Since this function is odd, its full Fourier series has only sine terms (all of the cosine coefficients are zero), and because of the other symmetry, the even sine terms also drop out.

12.4 Pointwise convergence of Fourier series

1. The Fourier series of $f$ converges pointwise to the function $F$ defined by

$$F(x) = \begin{cases} 
  0, & x = \pm (2k - 1)\pi, \ k = 1, 2, 3, \ldots, \\
  (x - 2k\pi)^3, & (2k - 1)\pi < x < (2k + 1)\pi, \ k = 0, \pm 1, \pm 2, \ldots.
\end{cases}$$

3. There are many such functions $f$, but they all have one or more discontinuities. An example is

$$f(x) = \begin{cases} 
  x, & 0 \leq x \leq \frac{1}{2}, \\
  x + 1, & \frac{1}{2} < x \leq 1.
\end{cases}$$

5. If $\{f_N\}$ converges uniformly to $f$ on $[a, b]$, then it obviously converges pointwise (after all, the maximum difference, over $x \in [a, b]$, between $f_N(x)$ and $f(x)$ converges to zero, so each individual difference must converge to zero). To prove that uniform convergence implies mean-square convergence, define

$$M_N = \max\{|f(x) - f_N(x)| : a \leq x \leq b\}.$$ 

Then

$$\sqrt{\int_a^b |f(x) - f_N(x)|^2 \, dx} \leq \sqrt{\int_a^b M_N^2 \, dx} = M_N \sqrt{b - a}.$$ 

By hypothesis, $M_N \to 0$ as $N \to \infty$, which gives the desired result.
12.5 Uniform convergence of Fourier series

7. Suppose \( f : [0, \ell] \to \mathbb{R} \) is continuous. Then \( f_{\text{even}} \), the even, periodic extension of \( f \) to \( \mathbb{R} \), is defined by

\[
f_{\text{even}}(x) = \begin{cases} 
  f(x - 2k\ell), & 2k\ell \leq x < (2k + 1)\ell, \ k = 0, \pm 1, \pm 2, \ldots, \\ 
  f(-x + 2k\ell), & (2k - 1)\ell \leq x < 2k\ell, \ k = 0, \pm 1, \pm 2, \ldots. 
\end{cases}
\]

Obviously, then, \( f_{\text{even}} \) is continuous on every interval \((2k\ell, (2k + 1)\ell)\) and \(((2k - 1)\ell, 2k\ell)\), that is, except possibly at the points \( 2k\ell \) and \((2k - 1)\ell, k = 0, \pm 1, \pm 2, \ldots \). We have

\[
limit_{x \to 2k\ell^{-}} f_{\text{even}}(x) = \lim_{x \to 2k\ell^{-}} f(-x + 2k\ell) = \lim_{x \to 0^{+}} f(x) = f(0)
\]

and

\[
limit_{x \to 2k\ell^{+}} f_{\text{even}}(x) = \lim_{x \to 2k\ell^{+}} f(x - 2k\ell) = \lim_{x \to 0^{+}} f(x) = f(0).
\]

Since \( f_{\text{even}}(2k\ell) = f(0) \) by definition, this shows that \( f_{\text{even}} \) is continuous at \( x = 2k\ell \).

A similar calculation shows that

\[
\lim_{x \to (2k - 1)\ell^{-}} f_{\text{even}}(x) = \lim_{x \to (2k - 1)\ell^{+}} f_{\text{even}}(x) = f_{\text{even}}((2k - 1)\ell) = f(\ell),
\]

and hence that \( f_{\text{even}} \) is continuous at \( x = (2k - 1)\ell \).

9. Given \( f : [0, \ell] \to \mathbb{R} \), define \( \tilde{f} : [-2\ell, 2\ell] \to \mathbb{R} \) by

\[
\tilde{f}(x) = \begin{cases} 
  f(x), & 0 \leq x \leq \ell, \\ 
  f(2\ell - x), & \ell < x \leq 2\ell, \\ 
  -f(-x), & -\ell \leq x < 0, \\ 
  -f(x + 2\ell), & -2\ell \leq x < -\ell.
\end{cases}
\]

Then define \( F : \mathbb{R} \to \mathbb{R} \) to be the periodic extension of \( \tilde{f} \) to \( \mathbb{R} \). Assuming \( f \) is piecewise smooth, the quarter-wave sine series of \( f \) converges to \( F(x) \) if \( F \) is continuous at \( x \), and to

\[
\frac{1}{2} [F(x-) + F(x+)]
\]

if \( F \) has a jump discontinuity at \( x \). If \( f \) is continuous, then its quarter-wave sine series converges to \( f \) except possibly at \( x = 0 \). If \( f \) is continuous and \( f(0) = 0 \), then the quarter-wave sine series of \( f \) converges to \( f \) at every \( x \in [0, \ell] \).

12.5 Uniform convergence of Fourier series

1. The function \( h(x) \) fails to satisfy \( h(-1) = h(1) \), so its Fourier coefficients decay like \( 1/n \) and its Fourier series is the slowest to converge. The function \( f \) satisfies \( f(-1) = f(1) \), but \( df/dx \) has a discontinuity at \( x = 0 \) (and the derivative of \( f_{\text{pet}} \) also has discontinuities at \( x = \pm 1 \)). Therefore, the Fourier coefficients of \( f \) decay like \( 1/n^2 \). Finally, \( g_{\text{pet}} \) and its first derivative are continuous, but its second derivative has a jump discontinuity at \( x = \pm 1 \), so its Fourier coefficients decay like \( 1/n^3 \). The Fourier series of \( g \) is the fastest to converge.

3. Figure 12.6 shows the \( f \), its partial Fourier series with 21, 41, 81, and 161 terms, and the line \( y = 1.09 \). Zooming in near \( x = 0 \) shows that the overshoot is indeed about 9%; see Figure 12.7.

12.6 Mean-square convergence of Fourier series

1. (a) The infinite series

\[
\sum_{n=1}^{\infty} \frac{1}{n^2}
\]

converges to a finite value.\(^1\) Therefore, \( \{1/n\} \in \ell^2 \) and so, by Theorem 12.36, the sine series converges to a function in \( L^2(0, 1) \).

\(^1\) A standard result in the theory of infinite series is that

\[
\sum_{n=1}^{\infty} \frac{1}{n^k}
\]
Figure 12.6: The function \( f \) from Exercise 12.5.5, its partial Fourier series with 21, 41, 81, and 161 terms, and the line \( y = 1.09 \).

Figure 12.7: Zooming in on the overshoot in Figure 12.6.

(b) Figure 12.8 shows the sum of the first 100 terms of the sine series. The graph suggests that the limit \( f \) is of the form \( f(x) = m(1 - x) \). The Fourier sine series of such an \( f \) are

\[
c_n = 2 \int_0^1 m(1 - x) \sin(n\pi x) \, dx = \frac{2m}{n\pi}, \quad n = 1, 2, 3, \ldots
\]

Therefore, in order that \( c_n = 1/n \), we must have \( m = \pi/2 \). Therefore,

\[
f(x) = \frac{\pi}{2}(1 - x).
\]

Figure 12.8: The partial sine series, with 100 terms, from Exercise 12.6.1.

converges if \( k \) is greater than 1. This can be proved, for example, by comparison with the improper integral

\[
\int_1^\infty \frac{dx}{x^k}.
\]
3. The infinite series
\[ \sum_{n=1}^{\infty} \left( \frac{1}{\sqrt{n}} \right)^2 = \sum_{n=1}^{\infty} \frac{1}{n} \]
is the harmonic series, a standard example of a divergent series. Therefore, \( \{1/\sqrt{n}\} \notin \ell^2 \), and so the series does not converge to an \( L^2(0,1) \) function.

5. (a) The function \( f(x) = x^k \) belongs to \( L^2(0,1) \) if and only if the integral \( \int_0^1 x^{2k} \, dx \) is finite. Provided \( k \neq -1/2 \), we have
\[
\int_0^1 x^{2k} \, dx = \lim_{r \to 0+} \int_r^1 x^{2k} \, dx = \lim_{r \to 0+} \frac{1 - r^{2k+1}}{1 + 2k} = \begin{cases} \frac{1}{1+2k}, & k > -\frac{1}{2}, \\ \infty, & k < -\frac{1}{2}. \end{cases}
\]
For \( k = -1/2 \),
\[
\int_0^1 x^{2k} \, dx = \int_0^1 \frac{dx}{x} = \lim_{r \to 0+} \int_r^1 \frac{dx}{x} = \lim_{r \to 0+} -\ln(r) = \infty.
\]
Thus we see that \( f \in L^2(0,1) \) if and only if \( k > -1/2 \).

(b) The first few Fourier sine coefficients of \( f(x) = x^{-1/4} \), as computed by the DST, are approximately
\[ 1.5816, 0.25353, 0.06303, 0.01786, 0.004106, \ldots \]

(c) The graphs of \( f \), the partial Fourier sine series, with 63 terms, and the difference between the two, are shown in Figure 12.9.

![Figure 12.9](image)

Figure 12.9: The function \( f(x) = x^{-1/4} \), together with its partial Fourier sine series (first 63 terms) (top), and the difference between the two (bottom). See Exercise 12.6.5.

7. Since \( v_n \to v \), there exists a positive integer \( N \) such that
\[ n \geq N \Rightarrow \|v - v_n\| < \frac{\epsilon}{2}. \]
Then, if \( n, m \geq N \), we have
\[ \|v_n - v_m\| \leq \|v_n - v\| + \|v - v_m\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \]
Therefore, \( \{v_n\} \) is Cauchy.

12.7 A note about general eigenvalue problems

1. The proof is a direct calculation, using integration by parts (Green's identity): Suppose \( u, v \in C_D^2(\Omega) \). Then
\[
(K_D u, v) = -\int_{\Omega} \nabla \cdot (k(x) \nabla u) v = \int_{\Omega} k(x) \nabla u \cdot \nabla v - \int_{\partial \Omega} k(x) u \frac{\partial u}{\partial n}
\]
\[ = \int_{\Omega} k(x) \nabla u \cdot \nabla v \quad \text{(since } v = 0 \text{ on } \partial \Omega) \]
\[ = -\int_{\Omega} \nabla \cdot (k(x) \nabla v) u + \int_{\partial \Omega} k(x) u \frac{\partial v}{\partial n} \]
\[ = -\int_{\Omega} \nabla \cdot (k(x) \nabla v) u \quad \text{(since } u = 0 \text{ on } \partial \Omega) \]
\[ = (u, K_D v). \]
3. If \( t \) is very large, then we can approximate \( u(x,t) \) by the first term in its generalized Fourier series. Using the same notation as before for the eigenvalues and eigenfunctions of the negative Laplacian on \( \Omega \), we have

\[
u(x,t) \approx c_1 e^{-n\lambda_1 t/\rho c_1} \psi_1(x), \quad t \text{ large.}
\]

The constant \( c_1 \) is the first generalized Fourier coefficient of the initial temperature \( \bar{5} \):

\[
c_1 = \frac{\int_{\Omega} \bar{5} \psi_1}{\int_{\Omega} \psi_1^2}.
\]

The approximation is valid provided \( \lambda_1 \) is a \textit{simple} eigenvalue; that is, there is only one linearly independent eigenvector corresponding to \( \lambda_1 \). Then all of the other terms in the generalized Fourier series decay to zero much more rapidly than does the first term.
Chapter 13

More about Finite Element Methods

13.1 Implementation of finite element methods

1. We must check that the one-point rule gives the exact values for the integral $\int_{T^R} f$ when $f(\mathbf{x}) = 1$, $f(\mathbf{x}) = x_1$, or $f(\mathbf{x}) = x_2$. With $f(\mathbf{x}) = 1$, we have

$$
\int_{T^R} f = (\text{area of } T^R) \cdot 1 = \frac{1}{2}
$$

and

$$
\frac{1}{2} f \left( \frac{1}{3}, \frac{1}{3} \right) = \frac{1}{2}.
$$

With $f(\mathbf{x}) = x_1$, we have

$$
\int_{T^R} f = \int_0^1 \int_0^{1-x_1} x_1 \, dx_2 \, dx_1 = \int_0^1 (x_1 - x_1^2) \, dx_1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.
$$

On the other hand,

$$
\frac{1}{2} f \left( \frac{1}{3}, \frac{1}{3} \right) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6},
$$

so the rule is exact in this case as well. By symmetry, the rule must hold for $f(\mathbf{x}) = x_2$, which shows that the rule has degree of precision at least 1. To show that the degree of precision is not greater than 1, we note that, for $f(\mathbf{x}) = x_1^2$,

$$
\int_{T^R} f = \int_0^1 \int_0^{1-x_1} x_1^2 \, dx_2 \, dx_1 = \int_0^1 (x_1^2 - x_1^3) \, dx_1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12},
$$

while

$$
\frac{1}{2} f \left( \frac{1}{3}, \frac{1}{3} \right) = \frac{1}{2} \cdot \frac{1}{9} = \frac{1}{18}.
$$

3. We have

$$
\int_T f = \int_1^{3/2} \int_0^{2z_1 - 2} x_1 x_2^2 \, dx_2 \, dx_1 + \int_{3/2}^2 \int_0^{4-2z_1} x_1 x_2^2 \, dx_2 \, dx_1 = \frac{7}{120} + \frac{1}{15} = \frac{1}{8}.
$$

On the other hand, the mapping from $T^R$ to $T$ is

$$
x_1 = 1 + z_1 + \frac{1}{2} z_2,
$$

$$
x_2 = z_2.
$$

The Jacobian is

$$
J = \begin{bmatrix}
1 & \frac{1}{2} \\
0 & 1
\end{bmatrix}.
$$
and its determinant is 1. Therefore, with
\[ g(z_1, z_2) = f \left( 1 + z_1 + \frac{1}{2} z_2, z_2 \right) = \left( 1 + z_1 + \frac{1}{2} z_2 \right) z_2^2, \]
we have
\[ \int_T f = \int_{T^R} g = \int_0^1 \int_0^{1-z_1} \left( 1 + z_1 + \frac{1}{2} z_2 \right) z_2^2 dz_2 dz_1. \]
This last integral also equals 1/8.

5. Recall that the nodes in the mesh are enumerated from 1 to \( M \). Let the constrained boundary nodes be those with numbers \( c_1, c_2, \ldots, c_K \). We define a new \( K \times 1 \) array \( \text{CNodePtrs} \) whose \( i \)th entry is \( c_i \). (Thus \( \text{CNodePtrs} \) contains pointers into the \( \text{NodeList} \) array, allowing one to find the coordinates of any given constrained node.) We need the inverse of the mapping \( i \mapsto c_i \); let us define \( j \mapsto d_j \) by
\[ d_j = \begin{cases} i & \text{if } j = c_i, \\ 0 & \text{if } j \neq c_i \text{ for all } i = 1, 2, \ldots, K. \end{cases} \]
Now, under the original description of \( \text{NodePtrs} \), the \( i \)th entry is zero if the \( i \)th node is constrained. We now redefine the \( i \)th entry to be \( -d_i \), if the \( i \)th node, while the \( i \)th entry is still \( g_i \), if the \( i \)th node is free. (The use of the negative sign is just a trick to allow us to distinguish a free node from a constrained node; it avoids the need to store a separate flag.) Suppose we wish to solve a BVP with inhomogenous Dirichlet conditions (on all or part of the boundary). Let \( G \) be the piecewise linear function whose nodal value is zero everywhere except at the constrained boundary nodes. At the nonfree boundary nodes, the value of \( G \) is the assigned Dirichlet data. To solve the inhomogeneous Dirichlet problem, we merely need to modify the load vector by replacing
\[ F_i = \int_{\Omega} f \phi_i \]
by
\[ F_i = \int_{\Omega} f \phi_i - a(G, \phi_i). \]
The quantity \( a(G, \phi_i) \) is nonzero only if the free node \( n_i \), belongs to a triangle that also contains a constrained boundary node. The point here is simply that this can easily be determined from the information in the data structure. As we loop over the triangles, we can determine, for each, whether it contains both a free node and a constrained node. If it does, we modify the load vector accordingly.

### 13.2 Solving sparse linear systems

1. Computing \( A = LU \) requires \( n - 1 \) steps, and step \( i \) uses
\[ (n-i)(1+2(n-i)) = 2(n-i)^2 + (n-i) \]
arithmetic operations (this is determined by simply counting the operations in the pseudo-code on page 604).
The total number of operations required is
\[ \sum_{i=1}^{n-1} \left( 2(n-i)^2 + (n-i) \right) = 2 \sum_{j=1}^{n-1} j^2 + \sum_{j=1}^{n-1} j = \frac{2n^3}{3} - \frac{n^2}{2} - \frac{n}{6}, \]
This agrees with the \( O(2n^3/3) \) operation count given in the text.
The computation of \( L^{-1} b \) requires \( n - 1 \) steps, with \( 2(n-i) \) arithmetic operations per step. The total is
\[ 2 \sum_{i=1}^{n-1} (n-1) = 2 \sum_{j=1}^{n-1} j = n^2 - n. \]
The final step of back substitution requires \( n \) steps, with \( 2(n-i) + 1 \) operations per step, for a total of
\[ \sum_{i=1}^{n} \left( 2(n-i) + 1 \right) = 2 \sum_{j=1}^{n-1} j + n = n^2. \]
The total for the two triangular solves is \( 2n^2 - n \), which also agrees with the count given in the text.
3. With
\[ \phi(x) = \frac{1}{2} x \cdot Ax - b \cdot x, \]
we have
\[
\phi(x + y) = \frac{1}{2} (x + y) \cdot A(x + y) - b \cdot (x + y) = \frac{1}{2} x \cdot Ax + y \cdot Ax + \frac{1}{2} y \cdot Ay - b \cdot x - b \cdot y
= \phi(x) + (Ax - b) \cdot y + \frac{1}{2} y \cdot Ay.
\]

The fact that \( A \) is symmetric allows the simplification
\[
\frac{1}{2} x \cdot Ay + \frac{1}{2} y \cdot Ax = y \cdot Ax,
\]
which was used above. We thus see that
\[
\phi(x + y) = \phi(x) + \nabla \phi(x) \cdot y + O(\|y\|^2), \quad \text{with} \quad \nabla \phi(x) = Ax - b.
\]

5. An inner product has three properties:
   (a) \((x, x) \geq 0\) for all \(x\), and \((x, x) = 0\) if and only if \(x = 0\).
   (b) \((x, y) = (y, x)\) for all \(x, y\).
   (c) \((\alpha x + \beta y, z) = \alpha (x, z) + \beta (y, z)\) for all \(x, y, z\) and all \(\alpha, \beta\).

These properties hold for \((x, y)_A = x \cdot Ay\). The third property would be true for any matrix \(A\), the second property obviously requires that \(A\) be symmetric, and the first property requires that \(A\) be positive definite.

7. (a) With \(x^{(0)} = (4, 2)\), the negative gradient is \(-\nabla \phi(x^{(0)}) = (-2, -1)\), and so we must minimize
\[
\phi(x^{(0)} - \alpha \nabla \phi(x^{(0)})) = 7\alpha^2 - 5\alpha - 18.
\]

The minimizer is \(\alpha = 5/14\), and so
\[ x^{(1)} = \left( \frac{23}{7}, \frac{23}{14} \right). \]

(b) At \(x^{(1)} = (23/7, 23/14)\), the negative gradient is \((-3/14, 3/7)\), so we must minimize
\[
\phi(x^{(1)} - \alpha \nabla \phi(x^{(1)})) = \frac{27}{196} \alpha^2 + \frac{45}{196} \alpha - \frac{529}{28}.
\]

The minimizer is \(\alpha = 5/6\), and so
\[ x^{(2)} = \left( \frac{87}{28}, \frac{2}{28} \right). \]

The desired solution is \((3, 2)\).

(c) With \(x^{(0)} = 0\), the CG algorithm produces
\[ x^{(1)} \approx (2.67456, 2.34024), \]

with \(x^{(2)} = (3, 2)\), the exact solution.

9. If \(y = x - x^{(0)}\), then \(x = y + x^{(0)}\) and so
\[ Ax = b \Rightarrow Ay + Ax^{(0)} = b \Rightarrow Ay = b - Ax^{(0)}. \]

We can therefore replace \(b\) by \(b - Ax^{(0)}\), apply the CG algorithm to estimate \(y\), and add \(x^{(0)}\) to get the estimate of \(x\).
13.3 An outline of the convergence theory for finite element methods

1. Let the vector-valued function \( F \) be defined by \( F(x) = (f(x), 0) \). Then
\[
\nabla \cdot (\phi F) = \phi \nabla \cdot F + \nabla \phi \cdot F = \phi \frac{\partial f}{\partial x_1} + \frac{\partial \phi}{\partial x_1} f,
\]
so, by the divergence theorem,
\[
\int_{\Omega} \phi \frac{\partial f}{\partial x_1} + \int_{\Omega} \frac{\partial \phi}{\partial x_1} f = \int_{\Omega} \nabla \cdot (\phi F) = \int_{\Omega} \phi \mathbf{F} \cdot \mathbf{n} = \int_{\partial \Omega} \phi f n_1.
\]
If \( \phi \in C_C^\infty(\Omega) \), then the boundary integral vanishes, and we obtain
\[
\int_{\Omega} \phi \frac{\partial f}{\partial x_1} = -\int_{\Omega} \frac{\partial \phi}{\partial x_1} f.
\]
The derivation for \( \partial/\partial x_2 \) is exactly analogous.

3. A direct computation shows that
\[
\| f - g \|_{L^2} = \frac{1}{2} \text{ for all integers } m, n,
\]
while
\[
\| f - g \|_{H^1} = \frac{1}{2} \sqrt{1 + m^2 \pi^2 + n^2 \pi^2} \text{ for all } m, n.
\]
Thus the \( L^2 \) error is independent of \( m, n \), while the \( H^1 \) error becomes arbitrarily large as \( m, n \to \infty \).

5. The following table shows the \( L^2 \) and energy norm errors. The initial mesh has eight triangles, and the side lengths of the triangles are divided by two each time the mesh is refined.

<table>
<thead>
<tr>
<th>( h )</th>
<th>( L^2 ) error</th>
<th>energy error</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{2}/2 )</td>
<td>1.7803 \cdot 10^{-2}</td>
<td>1.0190 \cdot 10^{-1}</td>
</tr>
<tr>
<td>( \sqrt{2}/4 )</td>
<td>5.6705 \cdot 10^{-3}</td>
<td>5.3955 \cdot 10^{-2}</td>
</tr>
<tr>
<td>( \sqrt{2}/8 )</td>
<td>1.5112 \cdot 10^{-3}</td>
<td>2.7318 \cdot 10^{-2}</td>
</tr>
<tr>
<td>( \sqrt{2}/16 )</td>
<td>3.8399 \cdot 10^{-4}</td>
<td>1.3700 \cdot 10^{-2}</td>
</tr>
<tr>
<td>( \sqrt{2}/32 )</td>
<td>9.6392 \cdot 10^{-5}</td>
<td>6.8554 \cdot 10^{-3}</td>
</tr>
<tr>
<td>( \sqrt{2}/64 )</td>
<td>2.4123 \cdot 10^{-5}</td>
<td>3.4283 \cdot 10^{-3}</td>
</tr>
</tbody>
</table>

We see that the \( L^2 \) error goes down by a factor of approximately four each time the mesh is refined, which is consistent with \( O(h^2) \) convergence. Similarly, the energy norm of the error goes down by a factor of approximately two each time the mesh is refined, which is consistent with \( O(h) \) convergence.

13.4 Finite element methods for eigenvalue problems

1. (a) Let \( u, v \in \mathbb{R}^n \) contain the nodal values of piecewise linear functions \( u_h, v_h \), respectively, so that
\[
u_h = \sum_{i=1}^{n} u_i \phi_i, \quad v_h = \sum_{i=1}^{n} v_i \phi_i.
\]
It follows that
\[
(u_h, v_h) = \int_{\Omega} u_h v_h = \int_{\Omega} \sum_{i=1}^{n} (u_i \phi_i) \sum_{j=1}^{n} (v_j \phi_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \int_{\Omega} \phi_i \phi_j \right) u_i v_j = \sum_{i=1}^{n} \sum_{j=1}^{n} M_{ij} u_i v_j = u \cdot M v.
\]
(b) As shown in the text, if \( u \) and \( v \) are generalized eigenvectors for the problem \( K u = \lambda M u \), then \( L^T u \) and \( L^T v \) are orthogonal in the Euclidean norm. But then
\[
0 = L^T u \cdot L^T v = u \cdot LL^T v = u \cdot M v = (u_h, v_h).
\]
Thus \( u_h \) and \( v_h \) are orthogonal in the \( L^2 \) norm.

3. The smallest eigenvalue is approximately 9.80 (correct to three digits).