Lagrange multipliers and sensitivity

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The Lagrange multipliers for the NLP

$$\begin{aligned} \min \quad & f(x) \\ s.t. \quad & g(x) = 0 \end{aligned} \tag{1}$$

have an interpretation that I have not mentioned yet: In a certain sense, they quantify the change in the optimal value of (1) if the constraints are perturbed. To be precise, I define

$$p(u) = \min \{ f(x) \, : \, g(x) = u \} \, .$$

Provided x^* is a nonsingular point, the function p is defined on some neighborhood N of 0 in R^m. Moreover, there exist smooth functions $x : N \to \mathrm{R}^n$ and $\lambda : N \to \mathrm{R}^m$ such that $x(u)$ is the (locally) unique solution to the NLP defining p and $\lambda(u)$ is the corresponding Lagrange multiplier.[1]

Therefore

$$p(u) = f(x(u)) \tag{2}$$

and

$$g(x(u)) = u. \tag{3}$$

The desired result follows from differentiating (2) and (3) and combining the results. By the chain rule, differentiating (2) yields

$$\nabla p(u) = \nabla x(u) \nabla f(x(u))$$

and differentiating (3 yields

$$\nabla x(u) \nabla g(x(u)) = I.$$

In particular,

$$\begin{aligned} \nabla p(0) &= \nabla x(0) \nabla f(x^*), \\ \nabla x(0) \nabla g(x^*) &= I. \end{aligned}$$

Since $\nabla f(x^*) = \nabla g(x^*) \lambda^*$, the first equation yields

$$\begin{aligned} \nabla p(0) &= \nabla x(0) \left(\nabla g(x^*) \lambda^* \right) \\ &= \left(\nabla x(0) \nabla g(x^*) \right) \lambda^* \\ &= I \lambda^* \\ &= \lambda^*. \end{aligned}$$

This is the desired result:

$$\nabla p(0) = \lambda^*.$$

[1] This can be proved by applying the implicit function theorem to the system

$$\begin{aligned} \nabla f(x) - \nabla g(x) \lambda &= 0, \\ -g(x) + u &= 0. \end{aligned}$$

This shows that if the original constraints $g(x) = 0$ are perturbed to $g(x) = u$, where u is small, then the optimal value of the NLP is changed to approximately

$$\nabla p(0) \cdot u = \lambda^* \cdot u = \sum_{i=1}^{m} \lambda_i^* u_i.$$

Example 0.1 *As an explicit example of the above results, I will solve*

$$\begin{array}{ll} \min & f(x) \\ s.t. & g(x) = u, \end{array}$$

where $f : \mathrm{R}^3 \to \mathrm{R}$ *and* $g : \mathrm{R}^3 \to \mathrm{R}$ *are defined by* $f(x) = x_1 + x_2 + x_3$ *and* $g(x) = x_1^2 + x_2^2 + x_3^2 - 1$. *It is easy to show that the global minimizer of the NLP is*

$$x(\mu) = -\frac{\sqrt{1+u}}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and the corresponding Lagrange multiplier is

$$\lambda(\mu) = -\frac{\sqrt{3}}{2\sqrt{1+u}}$$

(provided $\mu > -1$*). In particular, with* $u = 0$*, the solution and Lagrange multiplier are*

$$x^* = -\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \lambda^* = -\frac{\sqrt{3}}{2}.$$

Then

$$p(u) = f(x(\mu)) = -\sqrt{3}\sqrt{1+u},$$

so

$$p'(u) = -\frac{\sqrt{3}}{2\sqrt{1+u}}$$

and, in particular,

$$p'(0) = -\frac{\sqrt{3}}{2} = \lambda^*,$$

as predicted.