MA5630
Problem Set 3

Due 2 April 2003

Instructions: Solve any three of the following problems. Turn in neatly written solutions. For computational problems, the use of MATLAB or *Mathematica* is recommended.

1. The purpose of this problem is to analyze the nonlinear program

$$\begin{array}{ll} \min & f(x) \\ s.t. & g(x) = 0 \end{array}$$

where $f : \mathrm{R}^3 \to \mathrm{R}$ and $g : \mathrm{R}^3 \to \mathrm{R}^2$ are defined by

$$\begin{aligned} f(x) &= (x_1 - 1)^2 + 2(x_2 - 2)^2 + 3(x_3 - 3)^2, \\ g(x) &= \left[\begin{array}{c} x_1 + x_2 + x_3 \\ x_3 - x_2^2 \end{array} \right]. \end{aligned}$$

   (a) Set up the system of equations representing the first-order necessary conditions and find all pairs (x^*, λ^*) satisfying this system.

   (b) For each solution (x^*, λ^*):

      i. Compute $\nabla g(x^*)$ and determine whether x^* is a regular point.

      ii. Find a basis for $\mathcal{N}(\nabla g(x^*)^T)$.

      iii. Compute $\nabla^2 \ell(x^*, \lambda^*)$ and determine whether it is positive definite, positive semidefinite, negative definite, or negative semidefinite on the subspace $\mathcal{N}(\nabla g(x^*)^T)$. Can you identify each solution as a local minimizer or local maximizer?

2. (a) Use Newton's method to solve the extended system for the NLP from Problem 1. Use the starting point $x^{(0)} = (-1, -1, -1)$, $\lambda^{(0)} = (-1, -1)$.

   (b) Use the quadratic penalty method to solve the NLP from Problem 1. Use the starting point $(-1, -1, -1)$ with $\mu = 1$ for the first minimization, and then decrease μ by a factor of 10 each time.

   (c) Use the augmented Lagrangian method to solve the NLP from Problem 1. Use the starting point $(-1, -1, -1)$ with initial Lagrange multiplier estimate $(-1, -1)$ and $\mu = 1$ for the first minimization. Is it necessary to decrease μ?

3. Let $Q(\cdot; \mu)$ be quadratic penalty function for the NLP

$$\begin{array}{ll} \min & f(x) \\ s.t. & g(x) = 0. \end{array}$$

Assume that x^* is a nonsingular point and a local minimizer of the NLP. For μ sufficiently small, let x^*_μ be a local minimizer of $Q(\cdot; \mu)$ and assume that $x^*_\mu \to x^*$ as $\mu \to 0$.

(a) Prove carefully that the condition number of $\nabla^2 Q(x_\mu^*;\mu)$ increases without bound as $\mu \to 0$. (Recall that the condition number of a symmetric matrix A is

$$\frac{|\hat{\lambda}_{\max}(A)|}{|\hat{\lambda}_{min}(A)|},$$

where $\hat{\lambda}_{max}(A)$ and $\hat{\lambda}_{\min}(A)$ denote the largest and smallest eigenvalues of A *in magnitude*.)

(b) Prove that x_μ^* can be computed by solving the system

$$\begin{aligned} \nabla f(x) - \nabla g(x)\lambda &= 0, \\ -g(x) - \mu\lambda &= 0. \end{aligned}$$

Does this system become increasingly ill-conditioned as $\mu \to 0$? (Note: The system is a nonlinear system, so its conditioning is determined by the condition number of the Jacobian at the solution. Here the Jacobian is symmetric and indefinite.)

4. Let f, g, Q, x^*, and x_μ^* be as in the previous problem. Suppose that for some $\hat{\mu} > 0$, $x_{\hat{\mu}}^* = x^*$.

   (a) Prove that $x_\mu^* = x^*$ for all $\mu \in (0, \hat{\mu})$.

   (b) Prove that x^* is a stationary point of f.

   (c) Prove that the Lagrange multiplier λ^* corresponding to x^* is the zero vector.

   (d) Does x^* have to be (unconstrained) local minimizer of f in this case? Prove it or give a counterexample.

5. Suppose x^* is a local minimizer and a nonsingular point for the NLP

$$\begin{aligned} \min \quad & f(x) \\ s.t. \quad & g(x) = 0. \end{aligned}$$

Let λ^* be the corresponding Lagrange multiplier. Does x^* have to be a local minimizer of the augmented Lagrangian $L(\cdot;\lambda^*;\mu)$ for *every* $\mu > 0$? Prove it or give a counterexample.