MA5630
Problem Set 1

Due 3 February 2003

Instructions: Solve any five of the following problems. Turn in neatly written solutions. If you do any of problems 6–8, also submit your code by email.
Note: For computational and programming problems, the use of MATLAB is recommended. Other software environments or languages (e.g. *Mathematica*, Fortran, C, etc.) can be used instead. If you choose to use Fortran or C, then you will have to find linear algebra software (e.g. LAPACK) for solving linear systems.

1. A subset S of R^n is called *convex* if

$$x^{(1)}, x^{(2)} \in S, \alpha_1, \alpha_2 \geq 0, \alpha_1 + \alpha_2 = 1 \;\Rightarrow\; \alpha_1 x^{(1)} + \alpha_2 x^{(2)} \in S.$$

   Suppose $f : \mathrm{R}^n \to \mathrm{R}$ is convex and has at least one global minimizer. Prove that the set of all global minimizers of f is a convex set.

2. Let $f : \mathrm{R}^n \to \mathrm{R}$ be differentiable and strictly convex. Prove that $\nabla f(x) = \nabla f(y)$ implies that $x = y$.

3. Let $f : \mathrm{R}^n \to \mathrm{R}$ be twice continuously differentiable. Prove that if there exists $y \in \mathrm{R}^n$ such that $\nabla^2 f(y)$ fails to be positive semidefinite, then f is not convex.

4. Suppose $F : \mathrm{R}^n \to \mathrm{R}^m$ is twice differentiable and $f : \mathrm{R}^n \to \mathrm{R}$ is defined by

$$f(x) = \frac{1}{2}\|F(x)\|^2 = \frac{1}{2}\sum_{i=1}^{n}(F_i(x))^2.$$

   Derive formulas for $\nabla f(x)$ and $\nabla^2 f(x)$. Use matrix-vector notation whenever possible.

5. Suppose $f : \mathrm{R}^n \to \mathrm{R}$ is twice differentiable and $A \in \mathrm{R}^{n\times n}$ is a nonsingular matrix. Define $\tilde{f} : \mathrm{R}^n \to \mathrm{R}$ by

$$\tilde{f}(\tilde{x}) = f(x), \text{ where } \tilde{x} = Ax.$$

   (The function $\tilde{f}$ is the representation of f in the variables $\tilde{x} = Ax$.)

   (a) Compute $\nabla \tilde{f}(\tilde{x})$ and $\nabla^2 \tilde{f}(\tilde{x})$ in terms of $\nabla f(x)$ and $\nabla^2 f(x)$.

   (b) Suppose that we wish to apply Newton's method to minimize f, starting from $x^{(0)}$. Then the next iterate is

$$x^{(1)} = x^{(0)} - \nabla^2 f(x^{(0)})^{-1} \nabla f(x^{(0)}).$$

   Show that if we applied Newton's method to $\tilde{f}$ beginning from $\tilde{x}^{(0)} = Ax^{(0)}$, the result is the same (albeit expressed in the transformed variables).

6. Write a program that performs Newton's method for solving $F(x) = 0$, where $F : \mathrm{R}^n \rightarrow \mathrm{R}^n$. The program should accept as input

- a starting vector $x^{(0)}$;
- a function that, given n and x, computes $F(x)$ and J, the Jacobian of F at x;
- an error tolerance ϵ;
- an interation limit N.

The program must compute the Newton iterates $x^{(1)}, x^{(2)}, \ldots$, stopping with $x^{(i)}$ satisfying either

$$\|x^{(i)} - x^{(i-1)}\| \leq \epsilon$$

or $i = N$.

Create two test problems, one in two variables and the other in three, to which you know solutions. Use your test problems to demonstrate that your program is working correctly.

7. Consider the system of nonlinear equations represented by $F(x) = 0$, where $F : \mathrm{R}^4 \rightarrow \mathrm{R}^4$ is defined by

$$F(x) = \begin{bmatrix} x_3 + x_4 - 2 \\ x_3 x_1 + x_4 x_2 \\ x_3 x_1^2 + x_4 x_2^2 - \frac{2}{3} \\ x_3 x_1^3 + x_4 x_2^3 \end{bmatrix}.$$

(a) Use your Newton's method program to find a solution to high accuracy. Write out all the iterates (to full precision). (This system is a special case of Problem 8. Consult the statement of Problem 8 for hints on choosing a starting point.)

(b) Make a table of the errors $\|x_i - x^*\|$. Do the error exhibit quadratic convergence as predicted by the theory?

8. An n-point quadrature rule for estimating

$$\int_{-1}^{1} f(x)\, dx$$

takes the form

$$\int_{-1}^{1} f(x)\, dx \doteq \sum_{i=1}^{n} w_i f(x_i),$$

where $x_1, x_2, \ldots, x_n \in (-1, 1)$ are the quadrature nodes and $w_1, w_2, \ldots, w_n$ are the quadrature weights. The *Gaussian quadrature* rules are derived by requiring that the n-point rule be exact for every f of the form $f(x) = x^i$, $i = 0, 1, 2, \ldots, 2n - 1$. In other words, the $2n$ parameters $x_1, x_2, \ldots, x_n, w_1, w_2, \ldots, w_n$ are chosen to satisfy the $2n$ equations

$$\sum_{j=1}^{n} w_j x_j^{(i-1)} = \int_{-1}^{1} x^{(i-1)}\, dx, \; i = 1, 2, \ldots, 2n.$$

Apply your Newton's method program to find the n-point rule for n as large as possible.

Hints:

(a) The quadrature nodes are known to be symmetric about $x = 0$, so that if $r \in (0, 1)$ is one node, then $-r$ is another node. Moreover, the weights for $\pm r$ are the same.

(b) The weights are known to be positive.

These facts can be used to choose good starting points. (Or, if you like, you can use these facts to reduce the number of unknowns and equations by a factor of 2.)

Warning: Newton's method will not converge on this particular problem unless it is given a good starting point. Some trial and error will probably be needed to find such a starting point.