Chapter 1

Some Model Partial Differential Equations

Finite element methods are flexible and powerful techniques for solving partial differential equations (PDEs). There are actually a number of methods that go under the name of finite elements, so it is somewhat misleading to refer, as I do in the title, to the finite element method. In this book, I describe in some detail the Galerkin finite element method for stationary (equilibrium) problems.

In the first part of the book, I derive the Galerkin finite element method, showing it to be the synthesis of three powerful ideas:

1. A boundary value problem can be transformed to an equivalent form, called the weak or variational form, that can be approached by different methods, both analytical and computational, than those that apply to the original form of the problem.

2. The Galerkin method produces the best approximation, from a given approximating subspace, to the true solution of a variational problem. Moreover, this best approximation is the solution to a finite-dimensional system of equations.

3. When the approximating subspace in the Galerkin method is chosen to be a subspace of piecewise polynomial functions, the resulting algorithm is both efficient and effective: the system of equations can be formed and solved efficiently even when the number of unknowns is very large, and the resulting approximate solution can be highly accurate.

The second part of the book addresses practical issues involved in implementing the finite element method in a computer program.

I begin by describing the classes of PDEs to which the finite element method will be applied.
1.1 Laplace’s equation; elliptic boundary value problems

Laplace’s equation is the partial differential equation

$$ -\Delta u = 0, \quad (1.1) $$

where the Laplace operator (or the Laplacian), $\Delta$, is defined by

$$ \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} $$
in two dimensions, or

$$ \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} $$
in three. Much of the analysis and many aspects of the numerical methods covered in this book are the same whether the equation is posed in two or three spatial dimensions, but for definiteness I will describe the two-dimensional case, with some comments about three-dimensional problems.

The inhomogeneous version of Laplace’s equation,

$$ -\Delta u = f, \quad (1.2) $$

where $f$ is a function defined on $\Omega$, is called Poisson’s equation.

The equations (1.1) and (1.2) are most commonly posed on a bounded domain $\Omega$ in $\mathbb{R}^2$. A domain is a connected open set. The word connected means that the set consists of only one “piece,” or, more precisely, that any two points in the set are joined by a curve lying entirely within the set. Open means that the boundary of the set is not a part of the set. Bounded means that the set is finite in extent, that is, that it can be enclosed by a circle with a finite radius. The boundary of $\Omega$ will be denoted by $\partial \Omega$, and the closure $\overline{\Omega}$ of $\Omega$ is the union of $\Omega$ and $\partial \Omega$.

The PDEs (1.1) and (1.2), by themselves, are insufficient to determine a unique solution. (Below I will point out, using physical reasoning, why this must be true.) Boundary conditions must be imposed to complete the specification of the problem. For example, if $f$ is a function defined on $\Omega$ and $g$ is a function defined on $\partial \Omega$, then

$$ -\Delta u = f \text{ in } \Omega, \quad (1.3) $$
$$ u = g \text{ on } \partial \Omega \quad (1.4) $$
is called a Dirichlet boundary value problem (BVP), and (1.4) is referred to as a Dirichlet boundary condition. A Neumann BVP has the form

$$ -\Delta u = f \text{ in } \Omega, \quad (1.5) $$
$$ \frac{\partial u}{\partial n} = h \text{ on } \partial \Omega, \quad (1.6) $$

where

$$ \frac{\partial u}{\partial n} $$
is the normal derivative of \( u \) on \( \partial \Omega \). If the vector \( n(x,y) \) is the outward-pointing normal vector to \( \partial \Omega \) at \((x,y) \in \partial \Omega\) and \( \nabla u \) is the gradient of \( u \),

\[
\nabla u(x,y) = \begin{bmatrix} \frac{\partial u}{\partial x}(x,y) \\
\frac{\partial u}{\partial y}(x,y) \end{bmatrix},
\]

then the normal derivative is defined by

\[
\frac{\partial u}{\partial n}(x,y) = \nabla u(x,y) \cdot n(x,y).
\]

As I explain below, a Dirichlet BVP for Laplace’s or Poisson’s equation has a unique solution (as long as the functions \( f \) and \( g \) are reasonable). The situation with the Neumann problem is more subtle: If the functions \( f \) and \( h \) are compatible, then the Neumann BVP has infinitely many solutions, any two of which differ by a constant. On the other hand, if \( f \) and \( g \) are not compatible, then there is no solution. I will explain the compatibility condition below on physical grounds and derive it in the next chapter.

### 1.1.1 Physical experiments modeled by Laplace’s equation

#### Steady-state heat flow

The first application of Laplace’s equation is to a flat metal plate occupying a domain \( \Omega \) in \( \mathbb{R}^2 \). The function \( u(x,y) \) represents the temperature at the point \((x,y) \in \Omega \). The plate is assumed to be insulated on the top and bottom, so heat can flow only in two dimensions. Such a plate has a third dimension, its thickness, but I will assume that neither the plate nor its temperature varies in the vertical direction, so that a two-dimensional model suffices.

Laplace’s equation,

\[ -\Delta u = 0 \text{ in } \Omega, \]

models the case of steady-state heat flow: the temperature is independent of time and the temperature gradient \( \nabla u \) indicates the flow of heat energy across the plate. Poisson’s equation,

\[ -\Delta u = f \text{ in } \Omega, \]

models steady-state heat flow in the plate, where the function \( f \) describes a heat source or sink in the plate. If \( f(x,y) > 0 \) for some \((x,y) \in \Omega\), then heat energy is being added at that point at a rate \( f(x,y) \) (in appropriate units). If \( f(x,y) < 0 \), then energy is being removed at \((x,y)\).

In this context, Dirichlet boundary conditions,

\[ u = g \text{ on } \partial \Omega, \]

indicate that the temperature of the plate is held fixed at the boundary, specifically, that the temperature at \((x,y) \in \partial \Omega\) is held fixed at \( g(x,y) \). The Dirichlet boundary value problem

\[
-\Delta u = 0 \text{ in } \Omega,
\]

\[ u = g \text{ on } \partial \Omega \]
models the following situation: The plate is insulated on the top and bottom, the
temperature at each point \((x, y)\) in the boundary is held fixed at the given value of
\(g(x, y)\), and the plate is allowed to reach equilibrium. The equilibrium temperature
distribution is then given by the solution \(u\) of the BVP.

Neumann boundary conditions,

\[
\frac{\partial u}{\partial n} = h \text{ on } \partial \Omega,
\]

indicate that the heat flux across the boundary is the prescribed value \(h\). The heat
flux is the flow of heat energy, in units of energy per time per unit length. In
particular, the homogeneous Neumann condition

\[
\frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega
\]

models the case of no heat flux—the boundary is insulated.

**Units and physical parameters**

The equations described above are nondimensional version of the PDEs; to describe
actual materials (such as an iron plate, for example) requires physical parameters.
In the heat flow problem described above, the relevant parameter is the *thermal con-
ductivity* \(\kappa\). The thermal conductivity is the constant of proportionality in Fourier’s
law, which postulates that the heat flux is proportional to the temperature gradient:

\[
\text{heat flux} = -\kappa \nabla u.
\]

The units of \(\kappa\) are energy per time per length per temperature. For example,
the thermal conductivity of iron near 0 degrees Celsius is \(\kappa = 0.836 \text{ W/}(\text{cm K})\).
Poisson’s equation is then properly written as

\[
-\kappa \Delta u = f \text{ in } \Omega.
\]

From this equation, the units of \(f\) can be determined. They must be the same as
the units of the left-hand side, which are energy per time per volume (for example,
\(\text{W/cm}^3\)).

The thermal conductivity \(\kappa\) is positive by definition. As I show in the next
chapter, the sign of \(\kappa\) also has an important mathematical significance. In that
chapter I will also show why I prefer to include the negative sign explicitly in
Laplace’s and Poisson’s equation.

If the material is heterogeneous, then thermal conductivity varies throughout
\(\Omega\): \(\kappa = \kappa(x, y)\). The PDE becomes more complicated:

\[
-\nabla \cdot (\kappa(x, y) \nabla u) = f(x, y) \text{ in } \Omega.
\]

The divergence operator, denoted \(\nabla\), is a partial differential operator that takes a
vector-valued function and produces a scalar-valued function as follows: if

\[
F(x, y) = \begin{bmatrix} F_1(x, y) \\ F_2(x, y) \end{bmatrix}
\]


then
\[ \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \]

The divergence of the gradient is the Laplacian:
\[ \nabla \cdot (\kappa \nabla u) = \nabla \cdot \left[ \frac{\partial u}{\partial x} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} \right] = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \]

Therefore, when \( \kappa \) is constant,
\[ -\nabla \cdot (\kappa \nabla u) = -\kappa \nabla \cdot \nabla u = -\kappa \Delta u. \]

The appropriate form of the Neumann boundary condition, taking into account the physical characteristics of the material, is
\[
\kappa \frac{\partial u}{\partial n} = h(x, y) \quad \text{on} \quad \partial \Omega. 
\tag{1.7}
\]

Since the heat flux, by Fourier’s law, is \( \kappa \nabla u \) and
\[ \kappa \frac{\partial u}{\partial n} = \kappa \nabla u \cdot n, \]
(1.7) simply says that the heat flux in the normal direction is the prescribed value \( h(x, y) \).

The Neumann BVP
\[
-\kappa \Delta u = f(x, y) \quad \text{in} \quad \Omega, \\
\kappa \frac{\partial u}{\partial n} = h(x, y) \quad \text{on} \quad \partial \Omega
\]
indicates that heat energy is being added to or taken away from the plate in two ways: in the interior (the effect of the heat source \( f \)) and across the boundary (the effect of the heat flux \( h \)). If the temperature \( u \) is in equilibrium, it must be the case that the net amount of heat added is zero. As I shall show in the next chapter, this is expressed by the compatibility condition that
\[
\int_{\Omega} f + \int_{\partial \Omega} h = 0.
\]
The first integral is the heat energy added to the interior, while the second is the amount entering across the boundary.

**Small vertical deflections of a membrane**

Another experiment modeled by Laplace’s or Poisson’s equation is the following: a membrane that occupies a domain \( \Omega \) when at rest is fixed along the boundary and subjected to a small transverse pressure. The point on the membrane originally at \( (x, y, 0) \), \( (x, y) \in \Omega \), moves to \( (x, y, u(x, y)) \) under the influence of the pressure.
Chapter 1. Some Model Partial Differential Equations

Of course, it is a simplifying assumption that the point moves only in the vertical direction. This is not exactly true, but it will be nearly true if the pressure is small enough.

Dirichlet conditions in this application indicate that the boundary of the membrane is fixed. For example,

\[ u = 0 \text{ on } \partial \Omega \]

means that the boundary is fixed in the original plane. An inhomogeneous Dirichlet condition, such as

\[ u = g \text{ on } \partial \Omega, \]

means that the membrane is stretched on a frame whose shape is determined by the boundary function \( g \).

In this context, a homogeneous Neumann boundary condition indicates that the boundary is free to move in the vertical direction. This condition is not very natural when applied to the entire membrane, but the following mixed boundary conditions would define an interesting BVP:

\[
\begin{align*}
  u &= 0 \text{ on } \Gamma_1, \\
  \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_2.
\end{align*}
\]

Here \( \Gamma_1 \) and \( \Gamma_2 \) form a partition of the boundary \( \partial \Omega \), and the boundary conditions indicate part of the boundary \( \Gamma_1 \) is fixed, while the remainder \( \Gamma_2 \) is free to move up and down.

Again, I have presented the nondimensional version of the equations. When taking into account the physical characteristics of the membrane, the relevant quantity is the tension \( T \) in the membrane, and Poisson’s equation takes the form

\[-T \Delta u = f \text{ in } \Omega.\]

A constant \( T \) means that the tension is the same throughout the membrane.

1.2 Other elliptic boundary value problems

Laplace’s equation is the prototypical elliptic PDE. An elliptic PDE describes certain equilibrium phenomena and has mathematical properties that will be described in the next chapter. Here I will simply give some more examples of elliptic PDEs.

1.2.1 The equations of isotropic elasticity

An elastic membrane is said to be isotropic if its elastic response is the same in every direction. This means that if the membrane is stretched by a certain traction, or rotated about a point and then stretched by the same traction, the response in the two experiments will be the same.

In two-dimensional linear elasticity, one models small planar deformations of an elastic membrane, and the unknown is the displacement of the material from a
1.2. Other elliptic boundary value problems

reference position. This displacement is a vector-valued quantity:

\[ u(x, y) = \begin{bmatrix} u_1(x, y) \\ u_2(x, y) \end{bmatrix}. \]

The displacement \( u \) has the following meaning: under the applied load, the point of the membrane originally at \((x, y)\) moves to the location \((x + u_1(x, y), y + u_2(x, y))\).

When the membrane is isotropic, it is described by two scalar quantities called the Lamé moduli, \( \mu \) and \( \lambda \).\(^4\) The Lamé moduli are constants if the membrane is homogeneous and functions of \((x, y)\) if it is heterogeneous. Since there are two unknown functions, \( u_1 \) and \( u_2 \), there are naturally two PDEs that together model the stretching of the membrane under an applied load. These are usually written in vector form as follows:

\[ \begin{align*}
-\nabla \cdot \sigma &= f \text{ in } \Omega, \\
\sigma &= 2\mu \varepsilon + \lambda \text{tr} \varepsilon I, \\
\varepsilon &= \frac{1}{2} \left( \nabla u + \nabla u^T \right).
\end{align*} \]

I will now identify each term in these equations. The gradient of a vector-valued function \( u \) is

\[ \nabla u(x, y) = \begin{bmatrix} \frac{\partial u_1}{\partial x}(x, y) \\ \frac{\partial u_1}{\partial y}(x, y) \\ \frac{\partial u_2}{\partial x}(x, y) \\ \frac{\partial u_2}{\partial y}(x, y) \end{bmatrix} \]

(this is called the Jacobian matrix in other contexts). The quantity \( \varepsilon \) is the (linearized) strain tensor, a measure of the local deformation of the membrane. The trace of \( \varepsilon \), \( \text{tr} \varepsilon \), is the sum of the diagonal entries of \( \varepsilon \):

\[ \text{tr} \varepsilon = \varepsilon_{11} + \varepsilon_{22}. \]

The tensor \( \sigma \) is called the stress tensor. It measures the elastic response of the membrane to the deformation described by the strain. In two dimensions, \( \sigma \) has units of force per length; the units become force per area in three dimensions. The divergence of a tensor is the vector whose components are the divergences of the rows of the tensor:

\[ \nabla \cdot \sigma = \begin{bmatrix} \frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{21}}{\partial y} \\ \frac{\partial \sigma_{12}}{\partial x} + \frac{\partial \sigma_{22}}{\partial y} \end{bmatrix}. \]

The stress-strain relationship, expressed by (1.9), is sometimes called the constitutive hypothesis. It is an assumption about the response of the particular material. On the other hand, (1.8) is the balance law, which is the same for all materials. The function \( f \) in (1.8) is the applied load, in units of force per area. It might be more natural to express the body force in units of force per mass, in which case the right-hand side of (1.8) would be \( \rho f(x, y) \), where \( \rho \) is the density of the membrane, expressed in mass per area.

\(^4\)For those familiar with engineering mechanics, the Lamé moduli are essentially equivalent to the Young’s modulus \( E \) and Poisson’s ratio \( \nu \) in that there is a one-to-one mapping between \((\mu, \lambda)\) and \((E, \nu)\).
As an exercise to verify that the notation is understood, the reader can assume that \( \mu \) and \( \lambda \) are constants and check that (1.8-1.10) are equivalent to the two scalar PDEs
\[
(2\mu + \lambda) \frac{\partial^2 u_1}{\partial x^2} + \mu \frac{\partial^2 u_1}{\partial y^2} + (\mu + \lambda) \frac{\partial^2 u_2}{\partial y \partial x} = f_1, \tag{1.11}
\]
\[
(\mu + \lambda) \frac{\partial^2 u_1}{\partial y \partial x} + \mu \frac{\partial^2 u_2}{\partial x^2} + (2\mu + \lambda) \frac{\partial^2 u_2}{\partial y^2} = f_2. \tag{1.12}
\]
However, as I will show in later chapters, it is most convenient to work with the equations in vector form.

The right-hand side of PDE (1.8) represents a body force acting on the interior of the membrane. A body force is a force that acts at a distance, such as gravity or an electromagnetic force. Often, in a membrane problem, there is no body force. Typically the load is introduced by a traction (applied stress) on the boundary, which leads to the boundary condition
\[
\sigma n = h \text{ on } \partial \Omega.
\]
Frequently the traction is applied to only part of the boundary, while the remainder is fixed (that is, not allowed to move). Therefore, a natural BVP for a membrane has mixed boundary conditions:
\[
\begin{align*}
  u &= 0 \text{ on } \Gamma_1, \\
  \sigma n &= h \text{ on } \Gamma_2
\end{align*}
\]
(\( \Gamma_1 \) and \( \Gamma_2 \) form a partition of \( \partial \Omega \)). These boundary conditions model a simple experiment, in which part of the boundary is held motionless and the membrane is stretched by a traction applied to the rest of the boundary.

The equations of isotropic elasticity for a three-dimensional elastic solid also take the form (1.8-1.10) when written in vector form. In that case, the displacement \( u \) has three components and \( \nabla u, \epsilon, \) and \( \sigma \) are all \( 3 \times 3 \).

### 1.2.2 General linear elasticity

For an elastic membrane that is not assumed to be isotropic, the stress-strain relationship is more complicated. Assuming a linear relationship, there exists a 4-tensor \( A = A_{ijkl} \) such that
\[
\sigma = A \epsilon,
\]
that is,
\[
\sigma_{ij} = \sum_{k=1}^{2} \sum_{l=1}^{2} A_{ijkl} \epsilon_{kl}, \quad i, j = 1, 2.
\]
To be physically meaningful, the tensor \( A \) must satisfy the symmetry conditions
\[
A_{ijkl} = A_{jikl}, \quad i, j, k, l = 1, 2,
\]
\[
A_{ijkl} = A_{ijlk}, \quad i, j, k, l = 1, 2.
\]
as well as the condition

\[ \epsilon \cdot \mathcal{A} \epsilon \geq 0 \text{ for all symmetric } \epsilon. \]

The dot product of two 2-tensors \( \epsilon, \sigma \) is defined by

\[ \epsilon \cdot \sigma = \sum_{i=1}^{2} \sum_{j=1}^{2} \epsilon_{ij} \sigma_{ij}. \]

Mathematically, the stronger condition

\[ \epsilon \cdot \mathcal{A} \epsilon > 0 \text{ for all symmetric } \epsilon \neq 0 \]

is convenient, as I will show in the next chapter. Different choices of the tensor \( \mathcal{A} \) express different types of anisotropy.

The boundary conditions described above for an isotropic membrane have the same meanings for an anisotropic membrane. Once again, the equations for a three-dimensional elastic body have exactly the same form as for a two-dimensional elastic membrane, with the indices describing the tensors taking the values 1, 2, 3 instead of just 1, 2.

**Exercises**

1. Show that (1.8-1.10) are equivalent to (1.11-1.12) when \( \mu \) and \( \lambda \) are constant.