Preface

This introductory text on partial differential equations (PDEs) has several features that are not found in other texts at this level, including:

- equal emphasis on classical and modern techniques.
- the explicit use of the language and results of linear algebra.
- examples and exercises analyzing realistic experiments (with correct physical parameters and units).
- a recognition that mathematical software forms a part of the arsenal of both students and professional mathematicians.

In this preface, I will discuss these features and offer suggestions for getting the most out of the text.

Classical and modern techniques

Undergraduate courses on PDEs tend to focus on Fourier series methods and separation of variables. These techniques are still useful after two centuries because they offer a great deal of insight into those problems to which they apply. However, the subject of PDEs has driven much of the research in both pure and applied mathematics in the last century, and students ought to be exposed to some more modern techniques as well.

The limitation of the Fourier series technique is its restricted applicability: it can be used only for equations with constant coefficients and only on certain simple geometries. To complement the classical topic of Fourier series, I present the finite element method, a modern, powerful, and flexible approach to solving PDEs. Although many introductory texts include some discussion of finite elements (or finite differences, a competing computational methodology), the modern approach tends to receive less attention and a subordinate place in the exposition. In this text, I have put equal weight on Fourier series and finite elements.

Linear algebra

Both linear and nonlinear differential equations occur as models of physical phenomena of great importance in science and engineering. However, most introductory texts focus on linear equations, and mine is no exception. There are several reasons why this should be so. The study of PDEs is difficult, and it makes sense to begin with the simpler linear equations before moving on to the more difficult nonlinear equations. Moreover, linear equations are much better understood. Finally, much of what is known about nonlinear differential equations depends on the analysis of linear differential equations, so this material is prerequisite for moving on to nonlinear equations.

Because we focus on linear equations, linear algebra is extremely useful. Indeed, no discussion of Fourier series or finite element methods can be complete unless it puts the results in the proper linear algebraic framework. For example, both methods produce the best approximate solution from certain finite-dimensional subspaces, and the projection theorem is therefore central to both techniques. Symmetry is another key feature exploited by both methods.

While many texts de-emphasize the linear algebraic nature of the concepts and solution techniques, I have chosen to make it explicit. This decision, I believe, leads to a more cohesive course and a better preparation for future study. However, it presents certain challenges. Linear algebra does not seem to receive the attention it deserves in many engineering and science programs, and so many students will take a course based on this text without the "prerequisites." Therefore, I present a fairly complete overview of the necessary material in Chapter 3, *Essential Linear Algebra*.

Both faculty previewing this text and students taking a course from it will soon realize that there is too much material in Chapter 3 to cover thoroughly in the couple of weeks it can reasonably occupy in a semester course. From experience I know that conscientious students dislike moving so quickly through material that they cannot master it. However, one of the keys to using this text is to avoid getting bogged down in Chapter 3. Students should try to get from it the "big picture" and two essential ideas:

- How to compute a best approximation to a vector from a subspace, with and without an orthogonal basis (Section 3.4).
- How to solve a matrix-vector equation when the matrix is symmetric and its eigenvalues and eigenvectors are known (Section 3.5).

Having at least begun to grasp these ideas, students should move on to Chapter 4 even if some details are not clear. The concepts from linear algebra will become much clearer as they are used throughout the remainder of the text.[1]

I have taught this course several times using this approach, and, although students often find it frustrating at the beginning, the results seem to be good.

Realistic problems

The subject of PDEs is easier to grasp if one keeps in mind certain standard physical experiments modeled by the equations under consideration. I have used these

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[1] Also, Chapter 4 is much easier going than Chapter 3, a welcome contrast!

models to introduce the equations and to aid in understanding their solutions. The models also show, of course, that the subject of PDEs is worth studying!

To make the applications as meaningful as possible, I have included many examples and exercises posed in terms of meaningful experiments with realistic physical parameters.

Software

There exists powerful mathematical software that can be used to illuminate the material presented in this book. Computer software is useful for at least three reasons:

- It removes the need to do tedious computations that are necessary to compute solutions. Just as a calculator eliminates the need to use a table and interpolation to compute a logarithm, a computer algebra system can eliminate the need to perform integration by parts several times in order to evaluate an integral. With the more mechanical obstacles removed, there is more time to focus on concepts.

- Problems that simply cannot be solved (in a reasonable time) by hand can often be done with the assistance of a computer. This allows for more interesting assignments.

- Graphical capabilities allow students to visualize the results of their computations, improving understanding and interpretation.

I expect students to use a software package such as MATLAB, *Mathematica*, or *Maple* to reproduce the examples from the text and to solve the exercises.

I prefer not to introduce a particular software package in the text itself, for at least two reasons. The explanation of the features and usage of the software can detract from the mathematics. Also, if the book is based on a particular software package, then it can be difficult to use with a different package. For these reason, my text does not mention any software packages except in a few footnotes. However, since the use of software is, in my opinion, essential for a modern course, I have written tutorials for MATLAB, *Mathematica*, and *Maple* that explain the various capabilities of these programs that are relevant to this book. These tutorials appear on the accompanying CD.

Outline

The core material in this text is found in Chapters 5-7, which present Fourier series and finite element techniques for the three most important differential equations of mathematical physics: Laplace's equation, the heat equation, and the wave equation. Since the concepts themselves are hard enough, these chapters are restricted to problems in a single spatial dimension.

Several introductory chapters set the stage for this core. Chapter 1 briefly defines the basic terminology and notation that will be used in the text. Chapter

2 then derives the standard differential equations in one spatial dimension, in the process explaining the meaning of various physical parameters that appear in the equations and introducing the associated boundary conditions and initial conditions.

Chapter 3, which has already been discussed above, presents the concepts and techniques from linear algebra that will be used in subsequent chapters. I want to reiterate that perhaps the most important key to using this text effectively is to move through Chapter 3 expeditiously. The rudimentary understanding that students obtain in going through Chapter 3 will grow as the concepts are used in the rest of the book.

Chapter 4 presents the background material on ordinary differential equations that is needed in later chapters. This chapter is much easier than the previous one, because much of the material is review for many students. Only the last two sections, on numerical methods and stiff systems, are likely to be new. Although the chapter is entitled *Essential Ordinary Differential Equations*, Section 4.3 is not formally prerequisite for the rest of the book. I included this material to give students a foundation for understanding stiff systems of ODEs (particularly, the stiff system arising from the heat equation). Similarly, Runge-Kutta schemes and automatic step control are not strictly needed. However, understanding a little about variable step size methods is useful if one tries to apply an "off-the-shelf" routine to a stiff system.

Chapter 8 extends the models and techniques developed in the first part of the book to two spatial dimensions (with some brief discussions of three dimensions).

The last two chapters provide a more in-depth treatment of Fourier series (Chapter 9) and finite elements (Chapter 10). In addition to the standard theory of Fourier series, Chapter 9 shows how to use the fast Fourier transform to efficiently compute Fourier series solutions of the PDEs, explains the relationships among the various types of Fourier series, and discusses the extent to which the Fourier series method can be extended to complicated geometries and equations with nonconstant coefficients. Sections 9.4-9.6 present a careful mathematical treatment of the convergence of Fourier series, and have a different flavor from the remainder of the book. In particular, they are less suited for an audience of science and engineering students, and have been included as a reference for the curious student.

Chapter 10 gives some advice on implementing finite element computations, discusses the solution of the resulting sparse linear systems, and briefly outlines the convergence theory for finite element methods. It also shows how to use finite elements to solve general eigenvalue problems. The tutorials on the accompanying CD include programs implementing two-dimensional finite element methods, as described in Section 10.1, in each of the supported software packages (MATLAB, *Mathematica*, and *Maple*). The sections on sparse systems and the convergence theory are both little more than outlines, pointing the students toward more advanced concepts. Both of these topics, of course, could easily justify a dedicated semester-long course, and I had no intention of going into detail. I hope that the material on implementation of finite elements (in Section 10.1) will encourage some students to experiment with two-dimensional calculations, which are already too tedious to carry out by hand. This sort of information seems to be lacking from most books accessible to students at this level.

Possible course outlines

In a one-semester course (42-45 class hours), I typically cover Chapters 1-7 and part of Chapter 8. I touch only lightly on the material concerning Green's functions and the Dirac delta function (Sections 4.6, 6.6, and 7.4.1), and sometimes omit Section 6.3, but cover the remainder of Chapters 1-7 carefully.

If an instructor wishes to cover a significant part of the material in Chapters 8-10, an obvious place to save time is in Chapter 4. I would suggest covering the needed material on ODEs on a "just-in-time" basis in the course of Chapters 5-7. This will definitely save time, since my presentation in Chapter 4 is more detailed than is really necessary. Chapter 2 can be given as a reading assignment, particularly for the intended audience of science and engineering students, who will typically be comfortable with the physical parameters appearing in the differential equations.

Acknowledgments

This book began when I was visiting Rice University in 1998-1999 and taught a course using the lecture notes of Professor William W. Symes. To satisfy my personal predilections, I rewrote the notes significantly, and for the convenience of myself and my students, I typeset them in the form of a book, which was the first version of this text. Although the final result bears, in some ways, little resemblance to Symes's original notes, I am indebted to him for the idea of recasting the undergraduate PDE course in more modern terms. His example was the inspiration for this project, and I benefited from his advice throughout the writing process.

I am also indebted to the students who have suffered through courses taught from early version of this text. Many of them found errors, typographical and otherwise, that might otherwise have found their way into print.

I would like to thank Professors Gino Biondini, Yuji Kodoma, Robert Krasny, Yuan Lou, Fadil Santosa, and Paul Uhlig, all of whom read part or all of the text and offered helpful suggestions.

The various physical parameters used in the examples and exercises were derived (sometimes by interpolation) from tables in the *CRC Handbook of Chemistry and Physics* [35].

The graphs in this book were generated with MATLAB. For MATLAB product information, please contact:

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As mentioned above, the CD also supports the use of *Mathematica* and *Maple*. For *Mathematica* product information, contact:

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