Further results on incomplete tBDs.

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- D.L. Kreher, R.S. Rees, A hole-size bound for incomplete t-wise balanced designs, *to appear in: the Journal of Combinatorial designs.*

- D.L. Kreher, and R.S. Rees, On the maximum size of a hole in an incomplete t-wise balanced design with specified minimum block size, *submitted to: OSU Mathematical Research Institute Monograph Series.*

- I. Adamczak and D.L. Kreher, ???, *in progress*

  `http://www.math.mtu.edu/~kreher/ABOUTME/preprints.html`

  `http://www.math.mtu.edu/~kreher/ABOUTME/talk.html`

Review

A t–*wise balanced design* (tBD) of type t–$(v, \mathcal{K}, \lambda)$ is a pair $(X, \mathcal{B})$

- X is a v–element set of *points*
- $\mathcal{B}$ is a collection of subsets of X called *blocks*
- $B \in \mathcal{B} \Rightarrow |B| \in \mathcal{K}$
- every t–element subset of X is in exactly λ blocks.

proper if $t < k < v$ for each $k \in \mathcal{K}$
Steiner if $\lambda = 1$

Example: A proper 2BD of type 2–$(9, \{3, 4\}, 4)$.

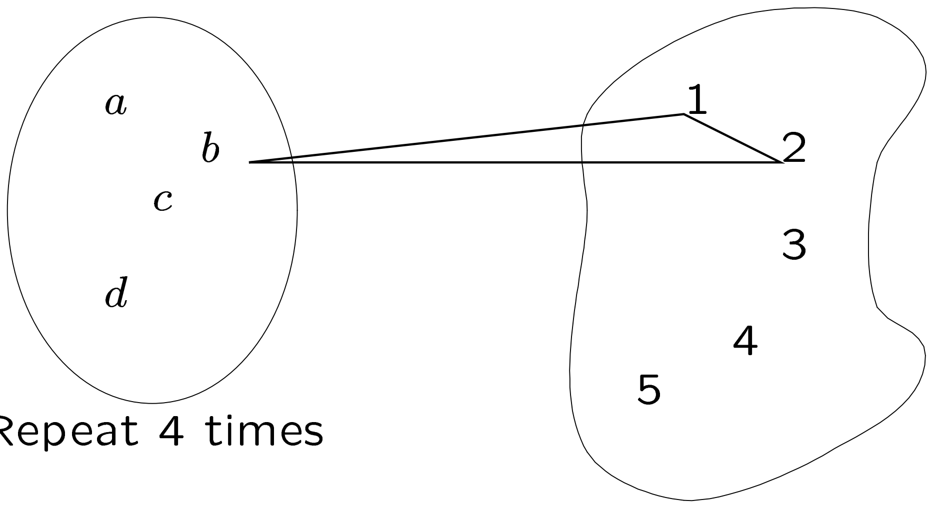
$$X = \{a, b, c, d, 1, 2, 3, 4, 5\}$$

$$\mathcal{B} = \left\{ \begin{array}{l} abcd, a12, a13, a14, a15, a23, a24, a25, a34, a35, a45, \\ abcd, b12, b13, b14, b15, b23, b24, b25, b34, b35, b45, \\ abcd, c12, c13, c14, c15, c23, c24, c25, c34, c35, c45, \\ abcd, d12, d13, d14, d15, d23, d24, d25, d34, d35, d45 \end{array} \right\}$$

$$\begin{aligned} \mathcal{B} = &\{xij : x \in \{a, b, c, d\}, i, j \in \{1, 2, \ldots, 5\}\} \\ &\cup \{abcd, abcd, abcd, abcd\} \end{aligned}$$

Example

- A 2BD of type 2–$(9, \{3, 4\}, 4)$



Repeat 4 times

History

1983 E.S. Kramer, Some results on t-wise balanced designs, Ars Combin. 15 (1983), 179–192.

- If B is a block in a Steiner tBD, then

$$|B| \leq (v+t-3)/2 \text{ for } t \geq 2.$$

- If B is a block in a Steiner tBD, then

$$\begin{aligned} |B| &\leq (v-1)/2 \text{ for } t = 2,4 \text{ while} \\ |B| &\leq v/2 \text{ for } t = 3,5. \end{aligned}$$

- **Conjecture:** If B is a block in a Steiner tBD, then

$$\begin{aligned} |B| &\leq (v-1)/2 \text{ for } t \text{ even while} \\ |B| &\leq v/2 \text{ for } t \text{ odd.} \end{aligned}$$

2000 M. Ira and E.S. Kramer, A block-size bound for Steiner 6-wise balanced designs, J. Combin. Designs, 8 (2000), 141-145.

- If B is a block in a Steiner 6BD, then

$$|B| \leq v/2.$$

ItBD

An *incomplete t–wise balanced design* (ItBD) of type t–$(v, h, \mathcal{K}, \lambda)$ is a triple $(X, H, \mathcal{B})$

- X is a v–element set of *points*
- H is an h–element set of points called the *hole*
- $\mathcal{B}$ is a collection of subsets of X called *blocks*
- $B \in \mathcal{B} \Rightarrow |B| \in \mathcal{K}$
- every t–element subset of X is either in the hole or in exactly λ blocks, but not both.

Example: A proper I2BD of type 2–$(9, 4, \{3\}, 4)$.

$$X = \{a, b, c, d, 1, 2, 3, 4, 5\}$$

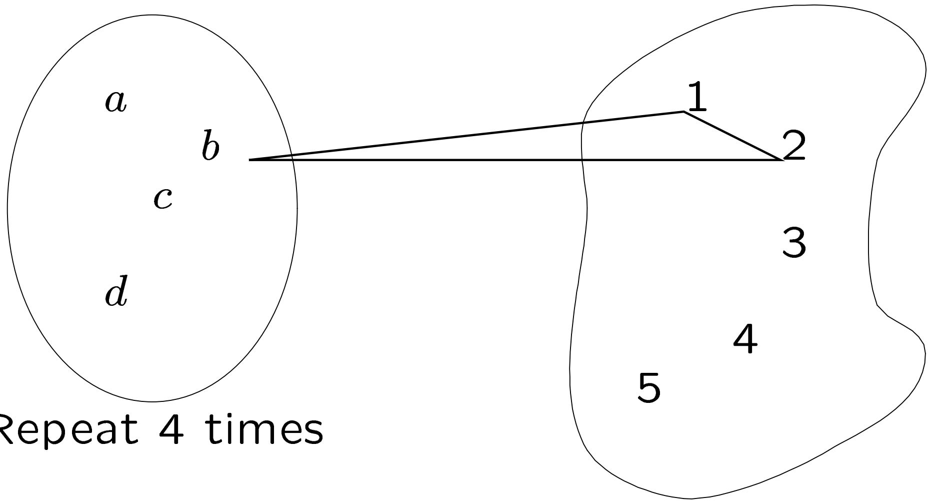
$$H = \{a, b, c, d\}$$

$$\mathcal{B} = \left\{ \begin{matrix} a12, a13, a14, a15, a23, a24, a25, a34, a35, a45, \\ b12, b13, b14, b15, b23, b24, b25, b34, b35, b45, \\ c12, c13, c14, c15, c23, c24, c25, c34, c35, c45, \\ d12, d13, d14, d15, d23, d24, d25, d34, d35, d45 \end{matrix} \right\}$$

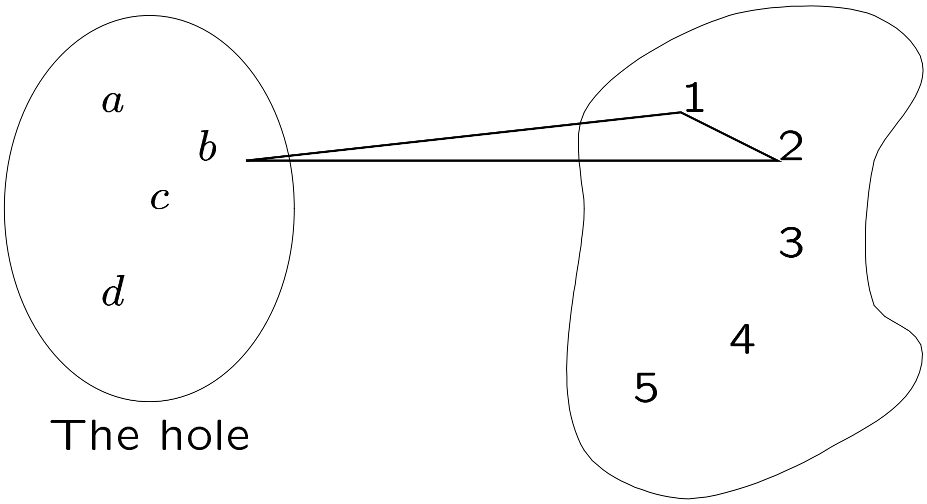
$$\mathcal{B} = \{xij : x \in \{a, b, c, d\}, i, j \in \{1, 2, \ldots, 5\}\}$$

Examples

- A 2BD of type 2–$(9, \{3,4\}, 4)$



- A I2BD of type 2–$(9, 4, \{3\}, 4)$



Equivalence

A ItBD of type t–$(v, h, \mathcal{K}, \lambda)$ is the same as

a tBD of type t–$(v, \mathcal{K} \cup \{h\}, \lambda)$
with a block of size h repeated λ times.

A Steiner tBD is a ItBD with $\lambda = 1$ in which any block of the tBD can be considered as the hole.

Main Theorem

If H is a hole in a ItBD with any λ, then

$$\begin{aligned} |H| &\leq (v-1)/2 \text{ for } t \text{ even while} \\ |H| &\leq v/2 \text{ for } t \text{ odd.} \end{aligned}$$

Corollary

If B is a block in a Steiner tBD, then

$$\begin{aligned} |B| &\leq (v-1)/2 \text{ for } t \text{ even while} \\ |B| &\leq v/2 \text{ for } t \text{ odd.} \end{aligned}$$

Outline of the proof:

1. **reduction to $\mathcal{K} = \{t+1\}$:**

   - If a proper ItBD $(X, H, \mathcal{B})$ of type t–$(v, h, \mathcal{K}, \lambda)$ exists, then a proper ItBD of type t–$(v, h, \{t+1\}, \lambda')$ exists for some λ'.

   Replace each block $B \in \mathcal{B}$ of size $|B| = k$ with its $\binom{k}{t+1}$ $(t+1)$-subsets and repeat each

   $$c_i = \prod_{j=1, j\neq i}^{\ell} (k_j - t) \text{ times.}$$

2. **Even suffices:**

   - If the main theorem is true when t is even, then it is also true when t is odd.

   The derived design of a $t-(v, h, \mathcal{K}, \lambda)$ with repect to a point in the hole is a $(t-1)$–$(v-1, h-1, \mathcal{K}-1, \lambda')$.

3. **Induction:** To show no ItBD with $h > \frac{v-1}{2}$.

   (a) **Basis:** There is no 2–$(v, h, 3, \lambda)$ with $h > \frac{v-1}{2}$.

   (b) **Derived design:**
   Takes care of $v = 2h-2, 2h-3, 2h-4, \ldots$

   (c) **Special case:** No ItBD with $v = 2h-1$.

   (d) **Special case:** No ItBD with $v = 2h$.

Bounds are sharp

1. For each odd $t \geq 3$ and each $h \geq t+1$ there is a
$$t\text{–}(2h, h, \{t+1\}, (2h-t)t!\binom{h-1}{t}).$$

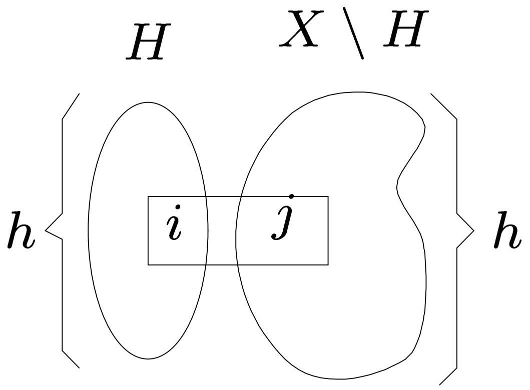
2. For each even $t \geq 2$ and each $h \geq t+1$ there is a
$$t\text{–}(2h+1, h, \{t+1\}, (2h-t+1)(t+1)!\binom{h}{t+1}).$$

For t odd we construct a t–$(2h, h, \{t+1\}, \lambda)$ with $\lambda = (2h - t)t!\binom{h-1}{t}$.

- Let $X = H \cup Y$, $|H| = |Y| = h$.
- Let $G = Sym(H) \times Sym(Y)$.
- Orbits of t sets are
$$\Delta_i = \{T \subseteq X : |T| = t \text{ and } |T \cap H| = i\}$$
For $i = 0, 1, 2, \ldots, t-1$. (The type $(j, t-j)$–sets.)
- Orbits of $t+1$ sets are
$$\Gamma_j = \{K \subseteq X : |K| = t+1 \text{ and } |K \cap H| = j\}$$
For $j = 0, 1, 2, \ldots, t-1$. (The type $(j, t+1-j)$–sets.)

Let A be the matrix given by
$$A[i,j] = |\{K \in \Gamma_j : K \supseteq T\}|$$
where $T \in \Delta_i$ is any fixed representative. A is square, upper-triangular and invertible. Thus there is a unique solution $\vec{x}$ to the matrix equation
$$A\vec{x} = \lambda J$$
where $\vec{x}$ is nonnegative and integral if $\lambda = (2h-t)t!\binom{h-1}{t}$.

The matrix



	x_0 $(0, t{+}1)$	x_1 $(1, t)$	x_2 $(2, t{-}1)$	x_3 $(3, t{-}2)$	$\cdots$ $\cdots$	x_{t-1} $(t{-}1, 2)$	λJ
$(0, t)$	$h{-}t$	h					$(2h{-}t)t!\binom{h-1}{t}$
$(1, t{-}1)$		$h{-}t{+}1$	$h{-}1$				$(2h{-}t)t!\binom{h-1}{t}$
$(2, t{-}2)$			$h{-}t{+}2$	$h{-}2$			$(2h{-}t)t!\binom{h-1}{t}$
$\vdots$				$\ddots$	$\ddots$		$(2h{-}t)t!\binom{h-1}{t}$
$(t{-}2, 2)$					$h{-}2$	$h{-}t{+}2$	$(2h{-}t)t!\binom{h-1}{t}$
$(t{-}1, 1)$						2	$(2h{-}t)t!\binom{h-1}{t}$

- Solve $A\vec{x} = \lambda J$.
- A is square, upper-triangular and invertible.
- Unique solution $\vec{x}$ to $A\vec{x} = \lambda J$
- $\lambda{=}(2h{-}t)t!\binom{h-1}{t} \Rightarrow \vec{x}$ is nonnegative and integral

Not all blocks have to be $(t+1)$-sets

The solution $\vec{x}$ to the equation $A\vec{x} = \lambda J$ was

$$x_{t-i} = t!\binom{h-1}{t}\left(1 + (-1)^{i-1}\frac{\binom{h-t+i}{i}}{\binom{h-1}{i}}\right).$$

$i = 1, 2, \ldots, t-1, t.$

Notice: the only $t+1$-sets that can cover type $(0,t)$ sets have type

$(0, t+1)$ i.e. orbit Γ_0
or
$(1, t)$ i.e. orbit Γ_1

When $i = t$ we have

$$x_0 = t!\left\{\binom{h-1}{t} + \binom{h}{t}\right\}.$$

When $i = t-1$ we have

$$x_1 = 0.$$

So the t-subsets of Y are covered by taking as blocks all of the $t+1$-subsets of Y each repeated x_0 times. Thus we can simply take Y as block and repeat it λ times. Thus we have a

$$t\text{–}(2h, h, \{t+1, h\}, (2h-t)t!\binom{h-1}{t})$$

Bounds are sharp for even t

For t even we construct a t-$(2h+1, h, \{t+1\}, \lambda)$ with $\lambda = (2h-t+1)(t+1)!\binom{h}{t+1})$.

Proof:

- We already know that there is a $\mathrm{I}(t+1)$BD $(H\dot{\cup}Y, H, \mathcal{B})$, of type

$$(t+1)\text{–}(2(h+1), h+1, \{t+2\}, \lambda')$$

  with $\lambda' = (2(h+1)-(t+1))(t+1)!\binom{(h+1)-1}{t+1})) = \lambda$.

- Take the derived design through a point $x \in H$ to get a

$$t\text{–}(2h+1, h, \{t+1\}, \lambda)$$

∎

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- Take the derived design through a point $x \in H$ to get a

  $$t\text{–}(2h+1, h, \{t+1\}, \lambda)$$

∎

Need $(t+1)$-sets

Consider a proper ItBD of type t–$(2h+1, h, \{t+2\}, \lambda)$ for t even and $h \geq t$. Let B be the matrix:

$$
\begin{array}{c|cccccc||c}
 & x_0 & x_1 & x_2 & x_3 & \cdots & x_{t-1} & \vec{t} \\
 & (0,t{+}2) & (1,t{+}1) & (2,t) & (3,t{-}1) & \cdots & (t{-}1,3) & \\
\hline
(0,t) & \binom{t+2}{2} & 1\cdot(t{+}1) & \binom{2}{2} & & & & \binom{h}{0}\binom{h+1}{t} \\
(1,t{-}1) & & \binom{t+1}{2} & 2\cdot t & \binom{3}{2} & & & \binom{h}{1}\binom{h+1}{t-1} \\
(2,t{-}2) & & & \binom{t}{2} & 3\cdot(t-1) & & & \binom{h}{2}\binom{h+1}{t-2} \\
\vdots & & & & \ddots & & \ddots & \\
(t{-}2,2) & & & & & & (t{-}1)\cdot 3 & \binom{h}{t-2}\binom{h+1}{2} \\
(t{-}1,1) & & & & & & \binom{3}{2} & \binom{h}{t-1}\binom{h+1}{t-1}
\end{array}
$$

Then there is a non-negative integer vector

$$\vec{x} = [x_0, x_1, \ldots, x_{t-1}]$$

satisfying: $\quad B\vec{x} = \lambda\vec{t}$

- $\vec{x} = \lambda A^{-1}\vec{t}$.
- $A^{-1}[1,j] = \frac{(-1)^{j-1}j}{\binom{t+2}{2}\binom{t}{j-1}}$
- $x_0 = \frac{-2\lambda t\binom{h+1}{t+1}}{(t+2)(2h-t)} < 0$ a contradiction.

Hence, no ItBD of the given type can exist.

Summary

There do not exist proper ItBDs of types

$$t\text{–}(2h+1, h, \mathcal{K}, \lambda) \qquad (t \text{ even})$$

or

$$t\text{–}(2h, h, \mathcal{K}, \lambda) \qquad (t \text{ odd})$$

for any λ and any $2 \le t \le h$,
with $\min\{k : k \in \mathcal{K}\} \ge t + 2$.

This raises the question:

What happens if we prescribe the minimum block size?

General bound

Theorem : *If* $(X, H, \mathcal{B})$ *is a proper* ItBD *of type* t–$(v, h, \mathcal{K}, \lambda)$ *with* $h \geq t \geq 2$ *and*

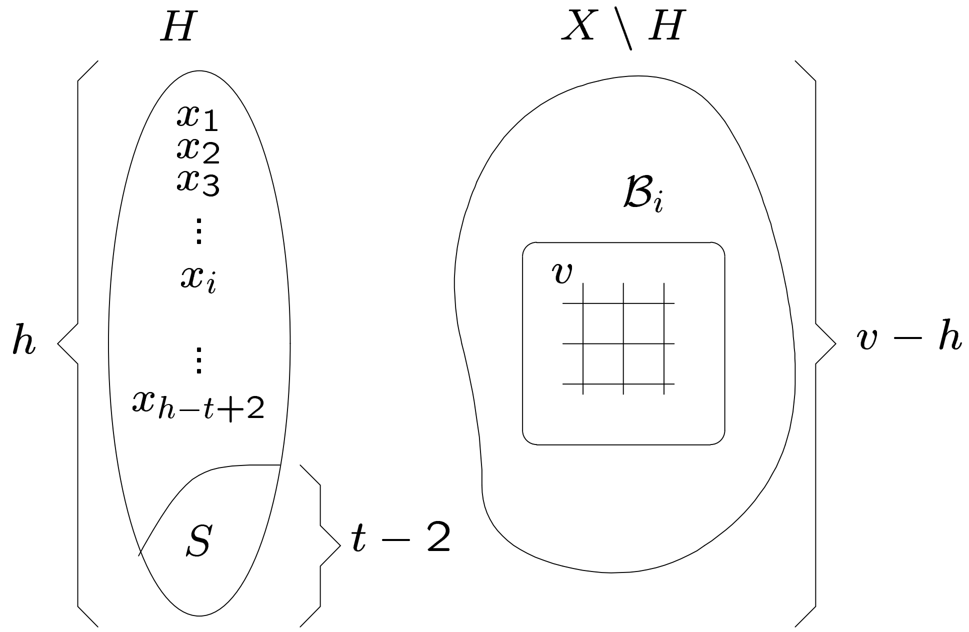
$$\min \mathcal{K} = k \geq t + 1,$$

then

$$h \leq \frac{v + (k-t)(t-2) - 1}{k - t + 1}.$$

These bounds are sharp for $t = 2$ and $t = 3$.

Proof of General bound



- $\mathcal{B}_i = \{B \setminus S \cup \{x_i\} : B \supseteq S \cup \{x_i\}\}$

  $\mathcal{B}_i$ is a λ-parallel class
  i.e. every $v \in X \setminus H$ is in λ blocks of $\mathcal{B}_i$

- We get $h-t+2$, λ parallel classes of blocks

  each block has size at least $k-(t-1) = k-t+1$.

- $\mathcal{B}_S = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \mathcal{B}_{h-t+2}$ (has repeated blocks)

1. Fix $v \in X \setminus H$.

2. Let N be the number of $(\{v, v'\}, B)$ where

   $v' \in X \setminus H$,

   $B \in \mathcal{B}_S$, and

   $v, v' \in B$.

3. $\{v, v'\}$ is in at most λ blocks of $\mathcal{B}_S$.
   So, $N \leq \lambda(v - h - 1)$.

4. On the other hand

   - v is in λ blocks of $\mathcal{B}_i$.
   - There are $h - t + 2$ $\mathcal{B}_i$s
   - Each block of $\mathcal{B}_i$ has size atleast $k - t + 1$

   So, $\lambda(h - t + 2)(k - t) \leq N$

Therefore:

$$\lambda(h - t + 2)(k - t) \leq \lambda(v - h - 1)$$

$$\begin{aligned}
(h - t + 2)(k - t) &\leq v - h - 1 \\
h(k - t) - (t - 2)(k - t) &\leq v - h - 1 \\
h(k - t + 1) - (t - 2)(k - t) &\leq v - 1 \\
h(k - t + 1) &\leq (t - 2)(k - t) + v - 1 \\
h &\leq \frac{v + (t - 2)(k - t) - 1}{k - t + 1}
\end{aligned}$$

Theorem : *For each $h \geq 3$ and each $k \geq 4$, $(h,k) \neq (3,4)$, there exists a* I3BD $(X, H, \mathcal{B})$ *of type* 3–$(v, h, \{k\}, \lambda)$ *where* $v = (k-2)h - (k-4)$ *and*

$$\lambda = \binom{v-3}{k-3}(k-1)!\binom{v-h-1}{k-1},$$

having $Sym(H) \times Sym(X \backslash H)$ *as an automorphism group.*

1. Three orbits $\Delta_0, \Delta_1, \Delta_2$ of 3–element

   Δ_i are the $(i, 3-i)$–sets.

2. There are three orbits $\Gamma_0, \Gamma_1, \Gamma_2$ of blocks available.

   Γ_j are the $(j, k-j)$–sets.

3. Let M be the 3 by 3 matrix

   $$M[i,j] = |\{B \in \Gamma_j : T \subseteq B\}|,$$

   where T is any fixed representative of Δ_i.

4. Then the design exists if and only if there is a non–negative integer vector $\vec{u}$ such that

   $$M\vec{u} = \lambda J,$$

$$M = \begin{bmatrix} \binom{v-h-3}{k-3} & h\binom{v-h-3}{k-4} & \binom{h}{2}\binom{v-h-3}{k-5} \\ 0 & \binom{v-h-2}{k-3} & (h-1)\binom{v-h-2}{k-4} \\ 0 & 0 & \binom{v-h-1}{k-3} \end{bmatrix}.$$

5. The sum along row i of M is

$$\binom{v-3}{k-3} - \sum_{\alpha=3-i}^{k-3} \binom{h-i}{\alpha}\binom{v-h-(3-i)}{k-3-\alpha}.$$

6. Let $\vec{v}$ be the unique solution to $M\vec{v} = \vec{w}$

$$\vec{w}_i = \sum_{\alpha=3-i}^{k-3} \binom{h-i}{\alpha}\binom{v-h-(3-i)}{k-3-\alpha}.$$

$$\begin{aligned} v_2 &= \frac{\binom{v-3}{k-3}}{\binom{v-h-1}{k-3}} - 1, \\ v_1 &= -1 \\ v_0 &= \frac{\binom{v-3}{k-3}}{\binom{v-h-3}{k-3}} \left\{ \frac{(h-2)(k-2)}{2(v-h-2)} \right\} - 1. \end{aligned}$$

7. $v_0 > -1$, $v_1 = -1$, and $v_2 > -1$.
   So, $\vec{v} + J$ is the non-negative rational solution to

$$M(\vec{v} + J) = \binom{v-3}{k-3} J.$$

8. Set $\vec{u} = (k-1)!\binom{v-h-1}{k-1}(\vec{v} + J)$. Then

$$M\vec{u} = \binom{v-3}{k-3}(k-1)!\binom{v-h-1}{k-1} J = \lambda J$$

   as desired.