FINAL REVIEW: KEY

1. Given the force field $\vec{F} = \vec{i} + 2\vec{j} - t\,\vec{k}$, find the work over the curve C given by the parametric equations

$$\begin{cases} x = 3\cos\sqrt{t} \\ y = 3\sin\sqrt{t} \\ z = 4\sqrt{t} \end{cases}$$

of a curve C, where $0 \leqslant t \leqslant \pi^2$.

Key:

$$W = \int 1\,dx + 2\,dy - t\,dz = \int_3^{-3} dx + \int_0^0 2\,dy - \int_0^{\pi^2} t\,d(4\sqrt{t})$$
$$= -6 + 0 - \int_0^{\pi^2} t\,\frac{4}{2\sqrt{t}}\,dt = -6 - \frac{4}{3}\pi^3.$$

2. For each of the following force fields that are path-independent, find the corresponding potential function:

(a) $\vec{F} = -y\,\vec{i} + x\,\vec{j}$; (b) $\vec{F} = (x^3 - 3xy^2, y^3 - 3x^2y)$.

Key:

(a) $\vec{F} = P\,\vec{i} + Q\,\vec{j}$, where $P = -y$ and $Q = x$. One has

$$\frac{\partial P}{\partial y} = -1, \quad \frac{\partial Q}{\partial x} = 1 \neq \frac{\partial P}{\partial y},$$

so that field $\vec{F}$ is not path-independent.

(b) $\vec{F} = P\,\vec{i} + Q\,\vec{j}$, where $P = x^3 - 3xy^2$ and $Q = y^3 - 3x^2y$. One has

$$\frac{\partial P}{\partial y} = -6xy, \quad \frac{\partial Q}{\partial x} = -6xy = \frac{\partial P}{\partial y},$$

so that field $\vec{F}$ is path-independent.

If $\vec{F} = \operatorname{grad} f$, then $f_x = x^3 - 3xy^2$ and $f_y = y^3 - 3x^2y$, so that the potential function is

$$\begin{aligned} f(x,y) &= \int f_x\,dx = \int (x^3 - 3xy^2)\,dx = \frac{x^4}{4} - \frac{3x^2y^2}{2} + g(y); \\ f_y &= -3x^2y + g'(y) = y^3 - 3x^2y; \\ g'(y) &= y^3; \\ g(y) &= \int y^3\,dy = \frac{y^4}{4} + C; \\ f(x,y) &= \frac{x^4}{4} - \frac{3x^2y^2}{2} + \frac{y^4}{4} + C = \frac{x^4 - 6x^2y^2 + y^4}{4} + C. \end{aligned}$$

3. For each of the force fields in Problem 2 that are not path-independent, find the work done over the circle of radius 2 centered at point $(0, 1)$. Do this in two ways: (i) using Green's theorem and (ii) without using it.

Key:

(i)

$$\int_R \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\, dA = \int_R (1 - (-1))\, dA = 2\,\text{Area}(R) = 2 \times \pi 2^2 = 8\pi.$$

(ii) Parametric eqs. are $x = 2\cos t$ and $y = 1 + 2\sin t$, $0 \leqslant t \leqslant 2\pi$.

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} = \int_C -y\, dx + x\, dy &= \int_0^{2\pi} -(1 + 2\sin t)(-2\sin t)\, dt + 2\cos t\,(2\cos t)\, dt \\ &= \int_0^{2\pi} (2\sin t + 4)\, dt = 8\pi.\end{aligned}$$

4. Find the following fluxes:

(a) of the field $\vec{F} = xyz\,\vec{i} + x\,\vec{j} - y\,\vec{k}$ through the square of side 2 centered on the x-axis in plane $x = 3$ and oriented in the negative x-direction;

(b) of the field $\vec{F} = -2\vec{r}$ through the sphere of radius 5 centered at the origin and oriented outward.

Key:

(a) $\vec{n} = -\vec{i}$;

$$\int_S (\vec{F} \cdot \vec{n})\, dA = \int_S (-xyz)\, dA = -3 \int_{-1}^{1} \int_{-1}^{1} yz\, dy\, dz = 0.$$

(b) $\vec{n} = \vec{r}/\|\vec{r}\| = \frac{1}{5}\,\vec{r}$;

$$\begin{aligned}\int_S (\vec{F} \cdot \vec{n})\, dA &= \int_S (-2\vec{r} \cdot (\tfrac{1}{5}\,\vec{r}))\, dA \\ &= -\frac{2}{5} \int_S \vec{r} \cdot \vec{r}\, dA \\ &= -\frac{2}{5} \times 5^2 \int_S dA = -\frac{2}{5} \times 5^2\,\text{Area}(S) = -\frac{2}{5} \times 5^2\, 4\pi \times 5^2 = -1000\pi.\end{aligned}$$

5. Use the Divergence Theorem to find the flux of the vector field $\vec{F} = (x^3, z, y)$ through the entire boundary surface (oriented outward) of the solid cylinder defined by the inequalities $x^2 + y^2 \leqslant 1$ and $-2 \leqslant z \leqslant 1$.

Key: $\operatorname{div} \vec{F} = \frac{\partial (x^3)}{\partial x} + \frac{\partial z}{\partial y} + \frac{\partial y}{\partial z} = 3x^2 + 0 + 0 = 3x^2$. So,

$$
\begin{aligned}
\int_S (\vec{F}\cdot\vec{n})\,dA &= \int_W \operatorname{div}\vec{F}\,dV \\
&= \int_W 3x^2\,dV \\
&= \int_0^{2\pi}\int_0^1\int_{-2}^1 3(r\cos\theta)^2 r\,dz\,dr\,d\theta \\
&=3(1-(-2))\int_0^{2\pi}\int_0^1 (r\cos\theta)^2 r\,dr\,d\theta \\
&=9\int_0^{2\pi}\int_0^1 (\cos\theta)^2 r^3\,dr\,d\theta \\
&=\frac{9}{4}\int_0^{2\pi}(\cos\theta)^2 d\theta = \frac{9}{4}\int_0^{2\pi}\frac{1}{2}\,(1+\cos 2\theta)d\theta = \frac{9}{4}\,\frac{1}{2}\,2\pi = \frac{9}{4}\,\pi.
\end{aligned}
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