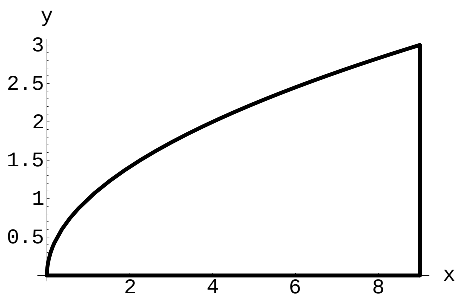
REVIEW 3: Key

1. Given the iterated integral $\int_0^3 \int_{y^2}^9 y\ \sin(x^2)\, dx\, dy$, draw the corresponding region of integration on the xy-plane, reverse the order of integration, and evaluate the integral.

Solution:



$$\begin{aligned}
&\int_0^3 \int_{y^2}^9 y\ \sin(x^2)\, dx\, dy \\
&= \int_0^9 \int_{y=0}^{y=\sqrt{x}} \sin(x^2)\, y\, dy\, dx \\
&= \tfrac{1}{2} \int_0^9 x\ \sin(x^2)\, dx \\
&= -\tfrac{1}{4} \cos(x^2)|_0^9 \\
&= \tfrac{1}{4} (1 - \cos 81) \approx 0.0558.
\end{aligned}$$

2. Find the volume under the graph of the function $f(x, y) = 3xy + y^2$ over the region R defined by the inequalities $x \geqslant 0$, $y \geqslant 0$, and $x + 2y \leqslant 2$.

Solution:

The region of integration is the triangle bounded by lines $x = 0$, $y = 0$, and $x + 2y = 2$.

$$\begin{aligned}
V &= \int_0^2 \int_{y=0}^{y=1-x/2} (3xy + y^2)\, dy\, dx \\
&= \int_0^2 \left(\frac{3}{2}\, xy^2 + \frac{y^3}{3} \right)\Bigg|_{y=0}^{y=1-x/2} dx \\
&= \int_0^2 \left(\frac{3}{2}\, x(1 - x/2)^2 + \frac{(1 - x/2)^3}{3} \right) dx \\
&= \int_0^2 \frac{1}{24} (4x^3 - 15x^2 + 12x + 4)\, dx = \frac{2}{3}.
\end{aligned}$$

3. Find the mass of the pyramid bounded by the planes $x = 0$, $y = 0$, $z = 1$, and $3x + 2y + z = 7$ given the mass density function $\delta(x, y, z) = 2z$.

Key:

$$\int_0^2 \int_0^{3-3x/2} \int_1^{7-3x-2y} 2z\, dz\, dy\, dx = 30.$$

4. Find the double integral

$$\int_{\pi/4}^{3\pi/4} \int_0^{4/\sin\theta} r\, dr\, d\theta$$

in polar coordinates and sketch the region of integration. Interpret the double integral geometrically as an area or volume.

Key:

$$\int_{\pi/4}^{3\pi/4} \int_0^{4/\sin\theta} r\,dr\,d\theta = \int_{\pi/4}^{3\pi/4} \frac{8}{\sin^2\theta}\,d\theta = 16.$$

The region is the triangle bounded by the lines $y = x$, $y = -x$, and $y = 4$. The double integral is the area of the region; it is also the volume of the solid between the planes $z = 0$ and $z = 1$ over the region.

5. Using cylindrical coordinates, find the mass of the upper half (above the xy-plane) of the unit ball $x^2 + y^2 + z^2 \leqslant 1$ given the mass density function $\delta(x, y, z) = z$. Find the same mass using spherical coordinates.

Key:

$$\int_0^{2\pi} \int_0^1 \left(\int_0^{\sqrt{1-r^2}} z\,dz \right) r\,dr\,d\theta = \frac{\pi}{4} = \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho\cos\varphi\,\rho^2\,\sin\varphi\,d\rho\,d\varphi\,d\theta.$$

6. Given the parametric equations

$$\begin{cases} x = 3\cos\sqrt{t} \\ y = 3\sin\sqrt{t} \\ z = 4\sqrt{t} \end{cases}$$

of a curve C, where $0 \leqslant t \leqslant \pi^2$, find the velocity and acceleration vectors as they depend on t. Find also the speed and the length of curve C. Describe the motion and its trajectory in words.

Key:

$$\begin{aligned} \vec{v} &= \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right) = \left(-\frac{3}{2\sqrt{t}} \sin\sqrt{t}, \frac{3}{2\sqrt{t}} \cos\sqrt{t}, \frac{2}{\sqrt{t}} \right); \\ \vec{a} &= \left(\frac{dv_x}{dt}, \frac{dv_y}{dt}, \frac{dv_z}{dt} \right) \\ &= \left(\frac{3}{4t\sqrt{t}} \sin\sqrt{t} - \frac{3}{4t} \cos\sqrt{t}, -\frac{3}{4t\sqrt{t}} \cos\sqrt{t} - \frac{3}{4t} \sin\sqrt{t}, -\frac{1}{t\sqrt{t}} \right); \\ \|\vec{v}\| &= \frac{5}{2\sqrt{t}}; \\ \text{length} &= \int_0^{\pi^2} \|\vec{v}\|\,dt = \int_0^{\pi^2} \frac{5}{2\sqrt{t}}\,dt = 5\pi. \end{aligned}$$

The motion starts at time $t = 0$ at point $(3, 0, 0)$ and proceeds along the helix trajectory with decreasing speed $\|\vec{v}\| = \dfrac{5}{2\sqrt{t}}$, to arrive at point $(-3, 0, 4\pi)$ at time $t = \pi^2$.