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Comparison of
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On the Bennett-Hoeffding inequality

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X_1, \dots, X_n : indep. r.v.'s s.t.

$X_i \leq y$ a.s. for some $y > 0$;

$E X_i \leq 0$;

$\sum_i E X_i^2 \leq \sigma^2$.

$S := X_1 + \dots + X_n$.

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$$f \in \mathcal{E} \stackrel{\text{def}}{\iff} \exists \lambda > 0 \forall u \in \mathbb{R} f(u) = e^{\lambda u};$$

$$f \in \mathcal{H}_+^\alpha \stackrel{\text{def}}{\iff} \exists \mu \geq 0 \forall u \in \mathbb{R} f(u) = \int_{-\infty}^{\infty} (u-t)_+^\alpha \mu(dt).$$

$$0 < \beta < \alpha \implies \mathcal{H}_+^\alpha \subseteq \mathcal{H}_+^\beta.$$

For $\alpha = 1, 2, \dots$: $f \in \mathcal{H}_+^\alpha$ iff $f^{(\alpha-1)}$ is convex and $f^{(j)}(-\infty) = 0$ for $j = 0, \dots, \alpha - 1$.

$$\bigcap_{\alpha > 0} \mathcal{H}_+^\alpha = \{f: f(x) = \int_{(0, \infty)} e^{tx} \mu(dt) \quad \forall x \in \mathbb{R}\}.$$

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Let Γ_{a^2} and Π_θ be any indep. r.v.'s s.t.

$$\Gamma_{a^2} \sim N(0, a^2) \text{ and } \Pi_\theta \sim \text{Pois}(\theta).$$

$$\tilde{\Pi}_\theta := \Pi_\theta - \mathbb{E} \Pi_\theta = \Pi_\theta - \theta.$$

Theorem (Main)

Take any $\beta > 0$ s.t.

$$\varepsilon := \frac{\beta}{\sigma^2 y} \in (0, 1).$$

Suppose that

$$\sum_i \mathbb{E}(X_i)_+^3 \leq \beta.$$

Then

$$\mathbb{E} f(S) \leq \mathbb{E} f(\Gamma_{(1-\varepsilon)\sigma^2} + y\tilde{\Pi}_{\varepsilon\sigma^2/y^2}) \quad \forall f \in \mathcal{H}_+^3.$$

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Proposition (Exactness for each f)

For each triple (σ, y, β) as in Theorem (Main) and each $f \in \mathcal{H}_+^3$, the upper bound $E f(\Gamma_{(1-\varepsilon)\sigma^2} + y\tilde{\Pi}_{\varepsilon\sigma^2/y^2})$ on $E f(S)$ is exact.

Proposition (Exactness in p)

For any given $p \in (0, 3)$, one cannot replace \mathcal{H}_+^3 in Theorem (Main) by the larger class \mathcal{H}_+^p .

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$$\forall f \in \mathcal{F} \quad \sup \mathbb{E} f(S) = \mathbb{E} f(\eta),$$

sup over all indep. X_i 's as before, with the cond.

$\sum \mathbb{E}(X_i)_+^3 \leq \beta$ imposed or not, and where the class \mathcal{F} of functions and the r.v. η are as in the following table:

Bound	\mathcal{F}	$\sum \mathbb{E}(X_i)_+^3 \leq \beta$ imposed?	η
BH	\mathcal{E}	no	$y\tilde{\Pi}_{\sigma^2/y^2}$
PU	\mathcal{E}	yes	$\Gamma_{(1-\varepsilon)\sigma^2} + y\tilde{\Pi}_{\varepsilon\sigma^2/y^2}$
Be	\mathcal{H}_+^2	no	$y\tilde{\Pi}_{\sigma^2/y^2}$
Pin	\mathcal{H}_+^3	yes	$\Gamma_{(1-\varepsilon)\sigma^2} + y\tilde{\Pi}_{\varepsilon\sigma^2/y^2}$

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Corollary (Upper bound on the tail)

Under the conditions of Theorem (Main), $\forall x \in \mathbb{R}$

$$\begin{aligned} P(S \geq x) &\leq \text{Pin}(x) := P_{\mathcal{H}_+^3}(\Gamma_{(1-\varepsilon)\sigma^2} + y\tilde{\Pi}_{\varepsilon\sigma^2/y^2}; x) \\ &\leq c_{3,0} P^{\text{LC}}(\Gamma_{(1-\varepsilon)\sigma^2} + y\tilde{\Pi}_{\varepsilon\sigma^2/y^2} \geq x), \end{aligned}$$

$$c_{3,0} = 2e^3/9 = 4.46\dots$$

Here, $P^{\text{LC}}(\eta \geq \cdot)$ is the least log-concave majorant of $P(\eta \geq \cdot)$, and

$$P_{\mathcal{F}}(\eta; x) = \inf \left\{ \frac{E f(\eta)}{f(x)} : f \in \mathcal{F}, f(x) > 0 \right\},$$

the best upper bound on $P(S \geq x)$ based on comparison $E f(S) \leq E f(\eta)$ for all $f \in \mathcal{F}$.

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Remark

Since the class \mathcal{H}_+^3 of generalized moment functs. is shift-invariant, it is enough to prove Theorem (Main) just for $n = 1$.

Fix any $\sigma > 0$ and $y > 0$.

For any $a \geq 0$ and $b > 0$, let $X_{a,b}$ denote any r.v. with the unique zero-mean distr. on the two-point set $\{-a, b\}$.

Lemma (Possible values of $E X_+^3$)

(i) For any r.v. X s.t. $X \leq y$ a.s., $E X \leq 0$, and $E X^2 \leq \sigma^2$,

$$E X_+^3 \leq \frac{y^3 \sigma^2}{y^2 + \sigma^2}.$$

(ii) For any

$$\beta \in \left(0, \frac{y^3 \sigma^2}{y^2 + \sigma^2}\right]$$

$\exists!$ $(a, b) \in (0, \infty) \times (0, \infty)$ s.t. $X_{a,b} \leq y$ a.s., $E X_{a,b}^2 = \sigma^2$,
and $E(X_{a,b})_+^3 = \beta$.

In particular, the ineq. in part (i) is exact.

Lemma (2-point zero-mean distrs. are extremal)

Fix any $w \in \mathbb{R}$, $y > 0$, $\sigma > 0$, and β s.t. $\beta \in \left(0, \frac{y^3 \sigma^2}{y^2 + \sigma^2}\right]$, and let (a, b) be the unique pair as in the previous lemma. Then

$$\begin{aligned} & \max\{E(X - w)_+^3 : X \leq y \text{ a.s.}, EX \leq 0, EX^2 \leq \sigma^2, EX_+^3 \leq \beta\} \\ &= \begin{cases} E(X_{a,b} - w)_+^3 & \text{if } w \leq 0, \\ E(X_{\tilde{a},\tilde{b}} - w)_+^3 & \text{if } w \geq 0, \end{cases} \end{aligned}$$

where $\tilde{b} := y$ and $\tilde{a} := \frac{\beta y}{y^3 - \beta}$. At that, $\tilde{a} > 0$, $X_{\tilde{a},\tilde{b}} \leq y$ a.s., $EX_{\tilde{a},\tilde{b}} = 0$, and $E(X_{\tilde{a},\tilde{b}})_+^3 = \beta$, but one can only say that $E X_{\tilde{a},\tilde{b}}^2 \leq \sigma^2$, and the latter inequality is strict if $\beta \neq \frac{y^3 \sigma^2}{y^2 + \sigma^2}$.

Lemma (Monotonicity in σ and β)

Take any $\sigma_0, \beta_0, \sigma, \beta$ s.t.

$$0 \leq \sigma_0 \leq \sigma, \quad 0 \leq \beta_0 \leq \beta,$$

$\beta_0 \leq \sigma_0^2 y$, and $\beta \leq \sigma^2 y$. Then

$$\mathbb{E} f(\Gamma_{\sigma_0^2 - \beta_0/y} + y \tilde{\Pi}_{\beta_0/y^3}) \leq \mathbb{E} f(\Gamma_{\sigma^2 - \beta/y} + y \tilde{\Pi}_{\beta/y^3}) \quad (1)$$

$\forall f \in \mathcal{H}_+^2$, and hence $\forall f \in \mathcal{H}_+^3$.

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Lemma (Main)

Let X be any r.v such that $X \leq y$ a.s., $\mathbb{E}X \leq 0$, $\mathbb{E}X^2 \leq \sigma^2$, and $\mathbb{E}X_+^3 \leq \beta$, where $\beta \in \left(0, \frac{y^3\sigma^2}{y^2+\sigma^2}\right]$. Then

$$\mathbb{E}f(X) \leq \mathbb{E}f(\Gamma_{\sigma^2-\beta/y} + y\tilde{\Pi}_{\beta/y^3}) \quad \forall f \in \mathcal{H}_+^3.$$

Sketch of proof By the “2-point zero-mean distrs. are extremal” lemma and the “monotonicity in σ and β ” lemma, w.l.o.g. $X = X_{a_0, b_0}$ for some $a_0 > 0$ and $b_0 > 0$. Also, w.l.o.g. $f(x) \equiv (x - w)_+^3$. Also, by rescaling, w.l.o.g. $y = 1$.

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The initial infinitesimal step:

Start with the r.v. X_{a_0, b_0} . Decrease a_0 and b_0 simultaneously by infinitesimal amounts $\Delta a > 0$ and $\Delta b > 0$ so that $E(X_{a_0, b_0} - w)_+^3 \leq E(X_{a, b} + X_{\Delta_1, \Delta_1} + X_{\Delta_2, 1} - w)_+^3 \forall w \in \mathbb{R}$, where $X_{a, b}, X_{\Delta_1, \Delta_1}, X_{\Delta_2, 1}$ are indep., $a = a_0 - \Delta a$ and $b = b_0 - \Delta b$, and $0 < \Delta_1 \approx 0$ and $0 < \Delta_2 \approx 0$ are chosen, together with Δa and Δb , so that to keep the balance of the total variance and that of the positive-part third moments closely enough:

$$E X_{a, b}^2 + E X_{\Delta_1, \Delta_1}^2 + E X_{\Delta_2, 1}^2 \approx E X_{a_0, b_0}^2$$

$$\text{and } E(X_{a, b})_+^3 + E(X_{\Delta_1, \Delta_1})_+^3 + E(X_{\Delta_2, 1})_+^3 \approx E(X_{a_0, b_0})_+^3.$$

Refer to X_{Δ_1, Δ_1} and $X_{\Delta_2, 1}$ as the symm. and highly asymm. infinitesimal spin-offs, resp.

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Continue decreasing a and b while “spinning off” the indep. pairs of indep. infinitesimal spin-offs X_{Δ_1, Δ_1} and $X_{\Delta_2, 1}$, at that keeping the balance of the total variance and that of the positive-part third moments, as described. Stop when $X_{a,b} = 0$ a.s., i.e., when a or b is decreased to 0 (if ever); such a termination point is indeed attainable. Then the sum of all the symm. indep. infinitesimal spin-offs X_{Δ_1, Δ_1} will have a centered Gaussian distr., while the sum of the highly asymmetric spin-offs $X_{\Delta_2, 1}$ ’s will give a centered Poisson component. At that, the balances of the variances and positive-part third moments will each be kept (the infinitesimal X_{Δ_1, Δ_1} ’s will provide in the limit a total zero contribution to the balance of the positive-part third moments).

Introduce a family of r.v.'s of the form

$$\eta_b := X_{a(b),b} + \xi_{\tau(b)} \quad \text{for } b \in [\varepsilon, b_0], \quad \text{where}$$

$$\varepsilon := \beta/\sigma^2 = b_0^2/(b_0 + a_0) < b_0,$$

$$a(b) := (b/\varepsilon - 1)b, \quad \tau(b) := a_0 b_0 - a(b)b, \quad (\text{balances})$$

$$\xi_t := W_{(1-\varepsilon)t} + \tilde{\Pi}_{\varepsilon t},$$

W . and $\tilde{\Pi}$. are indep. standard Wiener and centered standard Poisson processes, indep. of $X_{a(b),b}$ for each $b \in [\varepsilon, b_0]$. Note: $a(b_0) = a_0$ and $a(\varepsilon) = 0$, $\tau(b_0) = 0$ and $\tau(\varepsilon) = a_0 b_0 = \sigma^2$, so that

$$\eta_{b_0} = X_{a_0, b_0} \quad \text{and} \quad \eta_\varepsilon = W_{(1-\varepsilon)\sigma^2} + \tilde{\Pi}_{\varepsilon\sigma^2}.$$

Thus, it's enough to show that $E(\eta_b - w)_+^3$ decr. in $b \in [\varepsilon, b_0]$, for each $w \in \mathbb{R}$.

Proposition (PU(x) computation)

For all $\sigma > 0$, $y > 0$, $\varepsilon \in (0, 1)$, and $x \geq 0$

$$\begin{aligned} \text{PU}(x) &= e^{-\lambda_x x} \text{PU}_{\text{exp}}(\lambda_x) \\ &= \exp \frac{(1 - \varepsilon)^2 (w_x + 1)^2 - (\varepsilon + xy/\sigma^2)^2 - (1 - \varepsilon^2)}{2(1 - \varepsilon)y^2/\sigma^2}, \end{aligned}$$

$$\lambda_x := \frac{1}{y} \left(\frac{\varepsilon + xy/\sigma^2}{1 - \varepsilon} - w_x \right), \quad w_x := L \left(\frac{\varepsilon}{1 - \varepsilon} \exp \frac{\varepsilon + xy/\sigma^2}{1 - \varepsilon} \right),$$

and L is the Lambert product-log funct.: $\forall z \geq 0$, $w = L(z)$ is the only real root of the equation $we^w = z$.

Moreover, λ_x incr. in x from 0 to ∞ as x does so.

So, $\text{PU}(x)$ is about as easy to compute as $\text{BH}(x)$.

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Recall:

 $\text{Be}(x) := P_2(y\tilde{\Pi}_{\sigma^2/y^2}; x)$ and $\text{Pin}(x) := P_3(\Gamma_{(1-\varepsilon)\sigma^2} + y\tilde{\Pi}_{\varepsilon\sigma^2/y^2}; x)$, where

$$P_\alpha(\eta; x) := \inf_{t \in (-\infty, x)} \frac{E(\eta - t)_+^\alpha}{(x - t)^\alpha}.$$

An efficient procedure to compute $P_\alpha(\eta; x)$ in general was given in Pinelis '98.

In the case of $\text{Be}(x) = P_2(y\tilde{\Pi}_{\sigma^2/y^2}; x)$, this general procedure can be much simplified. Indeed, if α is natural and $\dots < d_k < d_{k+1} < \dots$ are the atoms of the distr. of η , then $E(\eta - t)_+^\alpha$ can be easily expressed for $t \in [d_k, d_{k+1})$ in terms of the truncated moments $E(\eta - d_k)_+^j$ with $j = 0, \dots, \alpha$.

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For $\text{Pin}(x) = P_3(\Gamma_{(1-\varepsilon)\sigma^2} + y\tilde{\Pi}_{\varepsilon\sigma^2/y^2}; x)$, there is no such nice localization property as for $\text{Be}(x) = P_2(y\tilde{\Pi}_{\sigma^2/y^2}; x)$, since the distr. of the r.v. $\Gamma_{(1-\varepsilon)\sigma^2} + y\tilde{\Pi}_{\varepsilon\sigma^2/y^2}$ is not discrete.

A good way to compute $\text{Pin}(x)$ turns out to be to express the positive-part moments $E(\eta - t)_+^\alpha$ for $\eta = \Gamma_{(1-\varepsilon)\sigma^2} + y\tilde{\Pi}_{\varepsilon\sigma^2/y^2}$ in terms of the Fourier or Fourier-Laplace transform of the distribution of η . Such expressions were developed in Pinelis '09 (with this specific motivation in mind). A reason for this approach to work is that the Fourier-Laplace transform of the distribution of the r.v. $\Gamma_{(1-\varepsilon)\sigma^2} + y\tilde{\Pi}_{\varepsilon\sigma^2/y^2}$ has a simple expression.

$$\mathbb{E} X_+^p = \frac{\Gamma(p+1)}{\pi} \int_0^\infty \Re e \frac{\mathbb{E} e_j((s+it)X)}{(s+it)^{p+1}} dt,$$

where $p \in (0, \infty)$, $s \in (0, \infty)$, Γ is the Gamma function, $\Re e z :=$ the real part of z , $i = \sqrt{-1}$, $j = -1, 0, \dots, \ell$, $\ell := \lceil p - 1 \rceil$, $e_j(u) := e^u - \sum_{m=0}^j \frac{u^m}{m!}$, and X is any r.v. s.t. $\mathbb{E} |X|^{j+} < \infty$ and $\mathbb{E} e^{sX} < \infty$.

Also,

$$\mathbb{E} X_+^p = \frac{\mathbb{E} X^k}{2} \mathbf{I}\{p \in \mathbb{N}\} + \frac{\Gamma(p+1)}{\pi} \int_0^\infty \Re e \frac{\mathbb{E} e_\ell(itX)}{(it)^{p+1}} dt,$$

where $k := \lfloor p \rfloor$ and X is any r.v. such that $\mathbb{E} |X|^p < \infty$. Of course, these formulas are to be applied here to $X = \Gamma_{(1-\varepsilon)\sigma^2} + y\tilde{\Pi}_{\varepsilon\sigma^2/y^2} - w$, $w \in \mathbb{R}$.

Compare the bounds BH, PU, Be, and Pin, and also the Cantelli bound

$$\text{Ca}(x) := \text{Ca}_{\sigma^2}(x) := \frac{\sigma^2}{\sigma^2 + x^2}$$

and the best exp. bound

$$\text{EN}(x) := \text{EN}_{\sigma^2}(x) \inf_{\lambda > 0} e^{-\lambda x} \mathbb{E} e^{\lambda \Gamma_{\sigma^2}} = \exp \left\{ -\frac{x^2}{2\sigma^2} \right\}$$

on the tail of $N(0, \sigma^2)$; of course, in general $\text{EN}(x)$ is *not* an upper bound on $P(S \geq x)$.

The bound $\text{Ca}(x)$ is optimal in its own terms.

Proposition

Take any $x \in [0, \infty)$, $\sigma \in (0, \infty)$, and r.v.'s ξ and η s.t. $\mathbb{E} \xi \leq 0 = \mathbb{E} \eta$ and $\mathbb{E} \xi^2 \leq \mathbb{E} \eta^2 = \sigma^2$. Then

$$P(\xi \geq x) \leq \text{Ca}(x) = \inf_{t \in (-\infty, x)} \frac{\mathbb{E}(\eta - t)^2}{(x - t)^2}.$$

Proposition

For all $x > 0$, $\sigma > 0$, $y > 0$, and $\varepsilon \in (0, 1)$,

- (I) $\text{Pin}(x) \leq \text{PU}(x) \leq \text{BH}(x)$ and $\text{Be}(x) \leq \text{Ca}(x) \wedge \text{BH}(x)$;
- (II) $\text{Be}(x) = \text{Ca}(x)$ for all $x \in [0, y]$;
- (III) $\text{BH}(x)$ increases from $\text{EN}(x)$ to 1 as y increases from 0 to ∞ ;
- (IV) $\exists u_{y/\sigma} \in (0, \infty)$ s.t. $\text{Ca}(x) < \text{BH}(x)$ if $x \in (0, \sigma u_{y/\sigma})$ and $\text{Ca}(x) > \text{BH}(x)$ if $x \in (\sigma u_{y/\sigma}, \infty)$; moreover, $u_{y/\sigma}$ incr. from $u_{0+} = 1.585\dots$ to ∞ as y/σ incr. from 0 to ∞ ; in particular, $\text{Ca}(x) < \text{EN}(x)$ if $x/\sigma \in (0, 1.585)$ and $\text{Ca}(x) > \text{EN}(x)$ for $x/\sigma \in (1.586, \infty)$.
- (V) $\text{PU}(x)$ incr. from $\text{EN}(x)$ to $\text{BH}(x)$ as ε incr. from 0 to 1.

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Proposition

For all $\sigma > 0$, $y > 0$, $\varepsilon \in (0, 1)$, and $x > 0$

$$\begin{aligned} \text{PU}(x) &= \max_{\alpha \in (0,1)} \text{EN}_{(1-\varepsilon)\sigma^2}((1-\alpha)x) \text{BH}_{\varepsilon\sigma^2, y}(\alpha x) \\ &= \text{EN}_{(1-\varepsilon)\sigma^2}((1-\alpha_x)x) \text{BH}_{\varepsilon\sigma^2, y}(\alpha_x x), \end{aligned}$$

where α_x is the only root in $(0, 1)$ of the equation

$$\frac{(1-\alpha)x^2}{(1-\varepsilon)\sigma^2} - \frac{x}{y} \ln \left(1 + \frac{\alpha xy}{\varepsilon\sigma^2} \right) = 0.$$

Moreover, α_x incr. from ε to 1 as x incr. from 0 to ∞ .

So, the bound $\text{PU}(x)$ is the product of the best exp. upper bounds on the tails $P(\Gamma_{(1-\varepsilon)\sigma^2} \geq (1-\alpha)x)$ and $P(\tilde{\Pi}_{\varepsilon\sigma^2} \geq \alpha x)$ — for some $\alpha \in (0, 1)$ (in fact, the $\alpha \in (\varepsilon, 1)$). This proposition is useful in establishing asymptotics of $\text{PU}(x)$.

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Proposition

For any fixed $\sigma > 0$, $y > 0$, and $\varepsilon \in (0, 1)$, and all $x \geq 0$

$$\text{Pin}(x) \leq \text{PU}(x) = (\varepsilon + o(1))^{x/y} \text{Be}(x) \leq (\varepsilon + o(1))^{x/y} \text{BH}(x)$$

as $x \rightarrow \infty$.

That is, for large x , the bound $\text{PU}(x)$ and, hence, the better bound $\text{Pin}(x)$ are each exponentially better than $\text{Be}(x)$ and hence than $\text{BH}(x)$ — especially when $\varepsilon \ll 1$.

Here, σ is normalized to be 1. In the next 4 frames, the graphs $G(P) := \{(x, \log_{10} \frac{P(x)}{\text{BH}(x)}) : 0 < x \leq x_{\max}\}$ for $P = \text{Ca}, \text{PU}, \text{Be}, \text{Pin}$, with the benchmark BH, will be shown, for $\varepsilon \in \{0.1, 0.9\}$, $y \in \{0.1, 1\}$, and $x_{\max} = 3$ or 4, depending on whether $y = 0.1$ (little skewed-to-the-right X_i 's) or $y = 1$ (much skewed-to-the-right X_i 's).

¶ for such choices of x_{\max} , the values of $\text{BH}(x_{\max}) \approx 0.016$ or 0.017, whether $y = 0.1$ or $y = 1$.

¶ $G(\text{Ca})$ is shown only on the interval $(0, u_y)$, on which $\text{Ca} < \text{BH}$, i.e., $\log_{10} \frac{\text{Ca}}{\text{BH}} < 0$.

¶ for $y = 1$, $\text{Ca}(x) < \text{BH}(x)$ for all $x \in (0, 2.66)$.

¶ For Pin, actually two approx. graphs are shown: the dashed and thin solid lines – produced using the Fourier-Laplace and Fourier formulas.

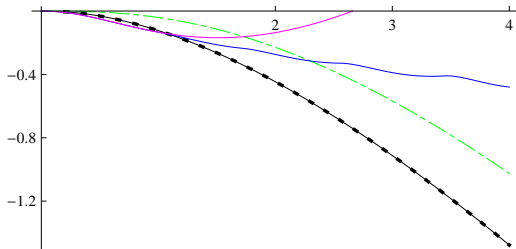
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Ca

Be

PU

Pin

If the weight of the Poisson component is small ($\varepsilon = 0.1$) and the Poisson component is quite distinct from the Gaussian component ($y = 1$), then $\text{Be}(x)$ is about 9.93 times worse (i.e., greater) than $\text{Pin}(x)$ at $x = 4$. Moreover, for these values of ε and y , even the bound $\text{PU}(x)$ is better than $\text{Be}(x)$ already at about $x = 2.5$.

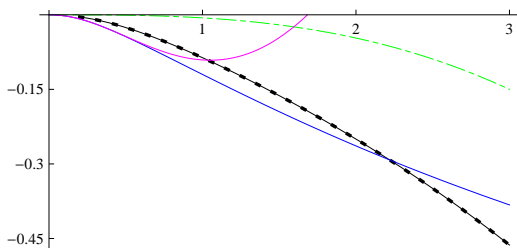
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Ca

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If the weight of the Poisson component is small ($\varepsilon = 0.1$) and the Poisson component is close to the Gaussian component ($y = 0.1$), then $\text{Be}(x)$ is still about 20% greater than $\text{Pin}(x)$ at $x = 3$.

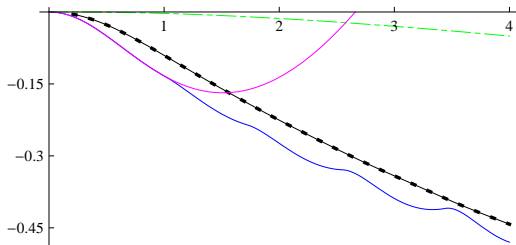
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Ca

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If the weight of the Poisson component is large ($\varepsilon = 0.9$) and the Poisson component is quite distinct from the Gaussian component ($y = 1$), then $\text{Be}(x)$ is about 8% better than $\text{Pin}(x)$ at $x = 4$. For $x \in [0, 4]$, $\text{Pin}(x)$ and $\text{Be}(x)$ are close to each other and both are significantly better than either $\text{BH}(x)$ or $\text{PU}(x)$ (which latter are also close to each other).

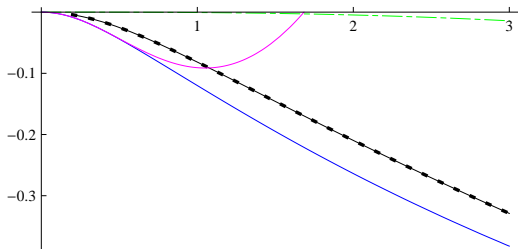
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PU

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If the weight of the Poisson component is large ($\varepsilon = 0.9$) and the Poisson component is close to the Gaussian component ($y = 0.1$), then $\text{Be}(x)$ is about 12% better than $\text{Pin}(x)$ at $x = 3$. For $x \in [0, 3]$, $\text{Pin}(x)$ and $\text{Be}(x)$ are close to each other and both are significantly better than either $\text{BH}(x)$ or $\text{PU}(x)$ (which latter are *very* close to each other).

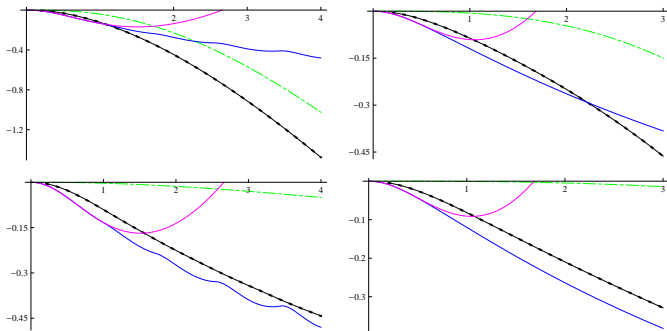
On the BH
ineq.

Pinelis

Outline

Introduction

Main results

Sketch of
proofComputation
of boundsComparison of
bounds

Row 1: $\varepsilon = 0.1$: heavy-tail Poisson component of little weight

Row 2: $\varepsilon = 0.9$: heavy-tail Poisson component of large weight

Column 1: $y = 1$: distrs. of the X_i 's may be much skewed to the right

Column 2: $y = 0.1$: distrs. of the X_i 's may be only a little skewed to the right.

On the BH
ineq.

Pinelis

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Thank you!