1 Introduction

Ratios are ubiquitous. A recent Google search for “ratio” returned over $96 \times 10^6$ items. Among those are numerous and commonly used financial ratios (visit, for example, http://www.onlinewbc.gov/docs/finance/fs_ratio1.html), as well as more exotic ones, such as “the ratio of Poisson’s ratio to the modulus of elasticity” [11] and “waist-to-hip ratio” [22].

Naturally, many ratios are varying (say, in time), so that they may be considered as the ratios $r := f/g$ of two functions, $f$ and $g$. One may then be interested in understanding the variability pattern of such a ratio $r$. In a number of models, the rates of change (that is, derivatives) $f'$ and $g'$ can be expressed simpler than $f$ and $g$, so that the variability pattern of the ratio $\rho := f'/g'$ of the rates of change may be easier to determine than the pattern of $r$. Then one may ask: What can be said in general about the variability pattern of the original ratio $r$ given the pattern of “derivative” ratio $\rho$? (Of course, $\rho$ is generally not the derivative of the ratio $r$ !)

In this connection, one would probably recall l’Hospital’s rule for limits. According to a history of mathematics by Nicholas Bourbaki (the pseudonym of a famous group of most prominent French mathematicians), calculus – the way we know it today – was forged within the 17th century, and Marquis Guillaume de l’Hospital (1661–1704) was one of a few most active mathematicians at the completion of it [6, page 178]. In fact, his book *L’Analyse...
des Infiniment Petits of 1696 was the first calculus book ever published. In that book, the marquis included what is commonly called l’Hospital’s rule (even though a now prevalent opinion is that the rule was communicated to him by Swiss mathematician Johann Bernoulli).

This famous rule is known to every student of calculus. It says that, under certain general and natural conditions, the limit of the ratio \( r = \frac{f}{g} \) of two functions \( f \) and \( g \) at a given point \( a \) is the same as the limit of the “derivative” ratio \( \rho = \frac{f’}{g’} \) at point \( a \) – provided that both functions \( f \) and \( g \) vanish (or go to infinity) at point \( a \). To illustrate this rule, imagine two cars starting on the same road at the same point at the same time. Then the l’Hospital rule says that at the very beginning of their trips the ratio of the distances of the two cars from their starting point will be close to the ratio of their speeds. In light of this interpretation, one would probably say that the rule is almost obvious. Yet, this rule is very powerful and of very broad use throughout mathematics and its applications.

Rather recently three separate groups of mathematicians discovered (independently, in different forms, and with different proofs) a similar rule – but for the monotonicity of the ratio rather than for its limit \([1, 7, 8, 12, 13, 19]\). They discovered the rule because they needed it, which already shows that the rule is useful. They discovered it independently because their respective areas of mathematics were quite distant from one another.

In the car setting, the new rule can be illustrated as follows: if the ratio \( \rho = \frac{f’}{g’} \) of the speeds of the two mentioned cars is increasing in time, then so is the ratio \( r = \frac{f}{g} \) of their distances from the starting point. Elucidated thus, this rule may seem intuitively obvious. However, it has found numerous useful applications: in analytic inequalities \([5, 13, 14, 17, 19]\), approximation theory \([5]\), differential and non-Euclidean geometry \([7–9, 18]\), information theory \([13, 14]\), mathematical physics \([1–4]\), statistics and probability \([12, 14–16, 21]\), etc.

Yet, in terms of the same car language, the range of applications may be further dramatically broadened if one can remove the restriction that the
cars start at the same point of the road. Suppose now that the two cars start at the same time at two possibly different points $A$ and $B$ on the same road, and the ratio of the cars’ distances from yet another point $C$ on that road is considered. The two cars may move in the same or opposite directions, but each of them keeps its own direction. The question is this: Given that the ratio of their speeds is monotonic in time, what can be said in general about the monotonicity pattern of the ratio of their distances from $C$ – say before either car arrives (if ever) at point $C$? The answer is $[14, 17]$ that then the ratio of the distances can switch at most once from the decreasing mode to the increasing one, or vice versa.

We refer to the latter rule as a general (l’Hospital-type) rule for monotonicity – as opposed to the special-case rule, valid under l’Hospital’s original condition that, in terms of the “car” language, both cars start at the same point at the same time, and their distances from the common starting point are considered.

It should be clear that the special-case rule for monotonicity is helpful wherever the classical l’Hospital rule for limits is so. However, the general rule reaches far beyond that, thanks to the start-at-the-same-point restriction having been removed. Recently, discrete-time analogues of the l’Hospital-type rules for monotonicity were obtained as well [20].

The main goal of this article is to try and make a case for the l’Hospital-type rules for monotonicity to be included into future calculus texts. It seems that here one has to answer the following two questions.

**Question 1:** Should these rules for monotonicity be in calculus texts, along with l’Hospital’s rules for limits? – While l’Hospital’s rules for limits take a prominent place in most calculus texts and are used very broadly throughout mathematics, the new rules for monotonicity are even more widely applicable. Indeed, as was explained in terms of the car language, now it is not required that the cars start at the same point at the same time. Also, in general the monotonicity patterns seem to carry no less information about a function than the values of limits. It follows that the answer to
Question 1 is: yes, certainly.

**Question 2:** Can these rules be reasonably presented in calculus texts?

– I believe that the answer here is also “yes, certainly.” Indeed, I will show that these rules are about as easy to state and prove and as much fun to use as l’Hospital’s rules for limits. I hope that the reader will agree that these new rules can be successfully studied even in the first semester of a calculus course.

The next section, Section 2, is thought of as a piece ready-to-include into a calculus text. The reader will probably notice that a few phrases from this Introduction are repeated there, for (it is hoped) understandable reasons; please accept my apologies for that. I have taken some liberty in the presentation given in Section 2; in particular, similarly to [10], I just say that the functions $f$ and $g$ are differentiable, rather than differentiable on an appropriate interval. Basic exercises, to be used mainly for homework assignments, are presented in Subsection 2.1. In Subsection 2.2, solutions to those exercises are provided. To save time and space, one may choose to include only the special-case rule into the course; however, then much (if not most) of the scope of applications and interesting mathematics would be lost. The textbook piece proposed in Section 2 would take up (including the homework exercises presented in Subsection 2.1 but not their solutions given in Subsection 2.2) about three pages in a standard textbook format, such as the one used in [10]. The latter textbook devotes over four pages to l’Hospital’s rule and its application, but it provides more examples – seven, versus the four examples given in Section 2 of this paper. On the other hand, Section 2 deals not only with the special-case rule for monotonicity (which roughly corresponds to l’Hospital’s rule for limits) but also with general rules.

More advanced exercises, appropriate for math majors, are collected in Section 3. Those exercises may be given in or outside the classroom. Of course, the set of examples presented in this paper can be significantly extended without too much effort.
2 L’Hospital-type rules for monotonicity

Ratios are ubiquitous. A recent Google search for “ratio” returned over $96 \times 10^6$ items. Among those are numerous and commonly used financial ratios, as well as more exotic ones, such as “waist-to-hip ratio.”

Naturally, many ratios are varying (say, in time), so that they may be considered as the ratios $r := f / g$ of two functions, $f$ and $g$. One may then be interested in understanding the variability pattern of such a ratio $r$. In a number of models, the rates of change (that is, derivatives) $f'$ and $g'$ can be expressed simpler than $f$ and $g$, so that the variability pattern of the ratio $\rho := f' / g'$ of the rates of change may be easier to determine than the pattern of $r$. (The Greek letter $\rho$ (rho) is read as row.) Then one may ask: What can be said in general about the variability pattern of the original ratio $r$ given the pattern of “derivative” ratio $\rho$? (Of course, $\rho$ is generally not the derivative of the ratio $r$!)

In this connection, one can recall l’Hospital’s rule for limits. Let $f$ and $g$ be differentiable functions, and let

$$r := \frac{f}{g} \quad \text{and} \quad \rho := \frac{f'}{g'}.$$ 

It is assumed throughout that $g$ and $g'$ do not take on the zero value and do not change their respective signs on some interval $(a, b)$ of the real line, so that both ratios, $r$ and $\rho$, are well defined on $(a, b)$. Here $a$ and $b$ can be either finite or infinite.

L’Hospital’s rule for limits says that, if $f(a) = g(a) = 0$, then

$$\lim_{x \to a} r(x) = \lim_{x \to a} \rho(x)$$

provided that the latter limit exists.

It turns out that there is a completely analogous rule, which allows one to infer the monotonicity pattern of ratio $r = f / g$ based on that of the derivative ratio $\rho = f' / g'$. Let us refer to this rule as “special-case,” as it is formulated for the special case when both $f$ and $g$ vanish at an endpoint of the interval $(a, b)$. 

5
Special-Case Rule for Monotonicity: Suppose that $f(a) = g(a) = 0$ or $f(b) = g(b) = 0$. Suppose also that $\rho \nearrow$ or $\searrow$ (that is, $\rho$ is increasing or decreasing) on the interval $(a, b)$. Then ratio $r$ has the same mononicity pattern on $(a, b)$ as $\rho$; that is, $r \nearrow$ or $\searrow$, respectively.

Example 1. For $x$ in either of the intervals $(0, 1)$ or $(1, \infty)$, consider the ratio
\[
\frac{r(x) = \ln x}{x - 1} = \frac{f(x)}{g(x)},
\]
where $f(x) := \ln x$ and $g(x) := x - 1$. Then $f(1) = 0$ and $g(1) = 0$ (so that $r(1)$ is not defined), and
\[
\rho(x) = \frac{f'(x)}{g'(x)} = \frac{(\ln x)'}{(x - 1)'} = \frac{1}{x} = \frac{1}{x} \searrow
\]
on $(0, 1)$ and on $(1, \infty)$. Now the special-case rule for monotonicity implies that $r \searrow$ on each of the intervals, $(0, 1)$ and $(1, \infty)$.

One can see that this rule works quite similarly to l'Hospital’s rule for limits, according to which one has
\[
\lim_{x \to 1} r(x) = \lim_{x \to 1} \frac{f(x)}{g(x)} = \lim_{x \to 1} \frac{f'(x)}{g'(x)} = \lim_{x \to 1} \rho(x)
= \lim_{x \to 1} \frac{1}{x} = 1.
\]
Thus, $r \searrow$ on the entire interval $(0, \infty)$ provided that $r(x)$ is extended to point $x = 1$ by continuity, so that $r(1) = \lim_{x \to 1} r(x) = 1$.

Recall that, to find the limits of some ratios, one may have to apply l'Hospital’s rule more than once. The same is true for the mononicity rules.

Example 2. For $x \neq 0$, consider the ratio
\[
r(x) = \frac{e^x - 1 - x}{x^2} = \frac{f(x)}{g(x)},
\]
where \( f(x) := e^x - 1 - x \) and \( g(x) := x^2 \). Then \( f(0) = 0 \) and \( g(0) = 0 \), and

\[
\rho(x) = \frac{f'(x)}{g'(x)} = \frac{e^x - 1}{2x} = \frac{f_1(x)}{g_1(x)},
\]

where \( f_1(x) := e^x - 1 \) and \( g_1(x) := 2x \). Then \( f_1(0) = 0 \) and \( g_1(0) = 0 \), and

\[
\frac{f_1'(x)}{g_1'(x)} = \frac{e^x}{2},
\]

and so, by the special-case rule, \( \rho = \frac{f_1}{g_1} \) and hence \( r \nearrow \) on \((-\infty, 0)\) and on \((0, \infty)\).

Again one can see a complete similarity with l’Hospital’s rule for limits, according to which

\[
\lim_{x \to 0} r(x) = \lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{f''(x)}{g''(x)} = \lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}.
\]

Thus, \( r \nearrow \) on the entire real line, \((-\infty, \infty)\), provided that \( r(x) \) is extended to point \( x = 0 \) by continuity, so that \( r(0) = \lim_{x \to 0} r(x) = \frac{1}{2} \).

We see that the special-case rule for monotonicity works wherever the l’Hospital rule for limits does. What is remarkable is that there are much more general rules for monotonicity, which work practically anywhere, even without the restriction that both \( f \) and \( g \) vanish at the same point \( a \).
General Rules for Monotonicity: If the derivative ratio $\rho = f'/g'$ is monotonic (that is, $\rho \nearrow$ or $\searrow$) on $(a,b)$, then the original ratio $r = f/g$ may switch from $\nearrow$ to $\searrow$ or vice versa at most once on $(a,b)$.

More specifically, the monotonicity pattern of ratio $r = f/g$ (on $(a,b)$) depends on that of ratio $\rho = f'/g'$ and on the sign of the ratio $g'/g$ according to this table:

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$g'/g$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nearrow$</td>
<td>$&gt; 0$</td>
<td>$\nearrow$ or $\searrow$ or $\nearrow$</td>
</tr>
<tr>
<td>$\searrow$</td>
<td>$&gt; 0$</td>
<td>$\nearrow$ or $\searrow$ or $\nearrow$</td>
</tr>
<tr>
<td>$\nearrow$</td>
<td>$&lt; 0$</td>
<td>$\nearrow$ or $\searrow$ or $\nearrow$</td>
</tr>
<tr>
<td>$\searrow$</td>
<td>$&lt; 0$</td>
<td>$\nearrow$ or $\searrow$ or $\nearrow$</td>
</tr>
</tbody>
</table>

Table 1: General rules for monotonicity.

Here, for instance, the pattern $r \searrow \nearrow$ means that $r$ switches from $\searrow$ to $\nearrow$ on $(a,b)$; that is, there is some $c$ in $(a,b)$ such that $r \searrow$ on $(a,c)$ and $\nearrow$ on $(c,b)$.

To discriminate between the three possible monotonicity patterns for $r$ in each row of Table 1, it is sufficient to know the monotonicity patterns of $r$ in a right neighborhood of $a$ and in a left neighborhood of $b$. Then Table 1 will determine the monotonicity pattern of $r$ on the entire interval $(a,b)$ without any ambiguity. Note that we say here “it is sufficient to know”, not “necessary”; there are other helpful tools which allow one to determine the monotonicity pattern of $r$ uniquely.

Let us see how the general rules for monotonicity work.

**Example 3.** For $x$ in the interval $(0, \frac{\pi}{2})$, consider the ratio

$$r(x) = \frac{-\frac{1}{2} + \sin x}{x} = \frac{f(x)}{g(x)},$$

where $f(x) := -\frac{1}{2} + \sin x$ and $g(x) := x$. Then $g(0) = 0$, but $f(0) = -\frac{1}{2} \neq 0$; also, $f(\frac{\pi}{2}) = \frac{1}{2} \neq 0$ and $g(\frac{\pi}{2}) = \frac{\pi}{2} \neq 0$. Therefore, the special-case rule for
monotonicity will not work for this ratio \( r \). However, the general rules will!

Indeed, here

\[
\rho(x) = \frac{f'(x)}{g'(x)} = \frac{\cos x}{1} = \cos x
\]

on \((0, \frac{\pi}{2})\). Also,

\[
g'(x)/g(x) = x'/x = 1/x > 0
\]

on \((0, \frac{\pi}{2})\). Thus, the 2nd line of Table 1 is applicable, so that \( r \rhd \) or \( \lambda \) or \( \smallfrown \) on \((0, \frac{\pi}{2})\). It remains to determine which of these three patterns actually takes place. To get a hint, let us plot the ratio \( r \).

\[
\text{It looks like } r \rhd \; \text{ but a picture is not quite a proof! Yet, it suggests that we can compute the values of } r(x) \text{ at appropriate trial points } x \text{ and thus exclude the monotonic patterns } r \rhd \text{ and } \lambda, \text{ whereupon } r \rhd \lambda \text{ will remain as the only possibility.}
\]

Equipped with all this knowledge, we can now compute, for instance,

\[
r\left(\frac{\pi}{6}\right) = 0, \quad r\left(\frac{3\pi}{8}\right) \approx 0.3598, \quad \text{and } r\left(\frac{\pi}{2}\right) = \frac{1}{\pi} \approx 0.3183.
\]

We see that

\[
r\left(\frac{\pi}{6}\right) < r\left(\frac{3\pi}{8}\right) > r\left(\frac{\pi}{2}\right), \quad \text{while } \frac{\pi}{6} < \frac{3\pi}{8} < \frac{\pi}{2},
\]

which does exclude both patterns \( \rhd \) and \( \lambda \) for \( r \). So, indeed the answer is \( r \rhd \lambda \)! Using the Mathematica command \texttt{FindMaximum\[r[x],\{x,3\Pi/8\}]}, we can also see that here the switch of ratio \( r \) from \( \rhd \) to \( \lambda \) on the interval \((0, \frac{\pi}{2})\) occurs at point \( x \approx 1.202 \) (which is fairly close to \( \frac{3\pi}{8} \approx 1.178 \)).

Now that we have seen how the rules for monotonicity work, we can ask the “why” question: Why do these rules work? In other words, is there a proof of these rules?

Suppose that the assumptions \( \rho \rhd \) and \( g'/g > 0 \) of the first line of Table 1 hold. For the sake of simplicity, let us also assume that the derivatives \( f' \) and \( g' \) are continuous and that \( r' \) has only finitely many (if any) roots in
It can be shown that one can do without these technical assumptions; anyway, in all our examples and exercises these additional assumptions are satisfied. Then it suffices to prove that $r'$ may change sign (if it ever does so) only from $-\rightarrow +$. To obtain a contradiction, suppose the contrary, that $r'$ changes sign from $+\rightarrow -$ at some point $u$ in $(a, b)$. Then $r'(u) = 0$ (by the continuity of $r'$), and in some right neighborhood $(u, u + h)$ of the root $u$ one has $r' < 0$ and hence $r < r(u)$. Consider now the key identity

$$r' = (\rho - r) g'/g,$$  \hspace{1cm} (1)

which is easy to check (Exercise 1). Then the conditions $r'(u) = 0$ and $r' < 0$ on $(u, u + h)$ imply, respectively, that $\rho(u) = r(u)$ and $\rho < r$ on $(u, u + h)$ (here we also use the condition $g'/g > 0$; see Exercise 2). It follows that $\rho < r < r(u) = \rho(u)$ on $(u, u + h)$, which contradicts the condition $\rho \not>$. The other three rows of Table 1 can be treated similarly.

The special case when both $f$ and $g$ vanish at the same endpoint of the interval $(a, b)$ is also easy to treat. For instance, suppose that $f(a) = g(a) = 0$ and $\rho \not>$. Then condition $g(a) = 0$ implies that $g$ and $g'$ must have the same sign on $(a, b)$. In view of the conditions $f(a) = g(a) = 0$ and the mean-value theorem (which we read “right-to-left”), for every $x$ in the interval $(a, x)$ there is some point $c$ in the interval $(a, x)$ such that

$$r(x) = \frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c)}{g'(c)} = \rho(c) < \rho(x),$$  \hspace{1cm} (2)

the latter inequality taking place because $c < x$ and $\rho \not>$. Therefore, for every $x$ in the interval $(a, b)$ one has $r(x) < \rho(x)$; that is, $\rho > r$ on $(a, b)$. Now it follows, in view of identity (1) and condition $g'/g > 0$, that $r' > 0$ and hence $r \not> (a, b)$. This verifies the special-case rule in the subcase when $f(a) = g(a) = 0$ and $\rho \not>$. The other three subcases, when $f(b) = g(b) = 0$ or $\rho \not<$, are treated similarly (Exercise 3).

The key identity (1) is illustrated by the following modeling example.

**Example 4.** Imagine a tank containing a solution of a chemical in water. Initially, the concentration (of the chemical in the solution) is 5%. More water
solution of the same chemical is continuously being added to the tank through a pipe, with the concentration of the chemical in the pipe steadily decreasing from 10% to 2%. What can be said about the monotonicity pattern of the concentration of the chemical in the tank?

The answer is as follows. Let \( f \) and \( g \) be, respectively, the current amount of the chemical in the tank and the current total amount of the solution there. Then, since both the total amount \( g \) of solution and its rate of change \( g' \) are positive, one has \( g'/g > 0 \). Also, some reflection (Exercise 5) shows that the current concentration of the chemical in the pipe coincides with \( \rho = f'/g' \), so that \( \rho \downarrow \). Therefore, by the second row of Table 1, the concentration \( r = f/g \) in the tank can switch at most once, and only from \( \nearrow \) to \( \searrow \). Moreover, initially the concentration in the pipe is \( \rho = 10\% \), which is greater than the initial concentration \( r = 5\% \) in the tank. It follows by identity (1) that initially one has \( r' > 0 \), and so, \( r \nearrow \) initially. If \( r \) were increasing all the way, then \( r \) would all the way be greater than its initial value of 5\%, while identity (1) would imply that \( \rho \geq r \) and hence \( \rho \geq 5\% \) all the way as well; but this would be a contradiction, because it is given that \( \rho = 2\% \) at the end of the process. This shows that \( r \) must switch exactly once, from the increasing mode to the decreasing one.

### 2.1 Exercises

1. Use the rule for differentiating a ratio (and also some algebra) to verify identity (1).

2. Given that \( g'/g > 0 \) on \( (u, u + h) \), verify that the conditions \( r'(u) = 0 \) and \( r' < 0 \) on \( (u, u + h) \) imply, respectively, that \( \rho(u) = r(u) \) and \( \rho < r \) on \( (u, u + h) \).

3. Verify the other three subcases of the special-case rule for monotonicity – when \( f(b) = g(b) = 0 \) or \( \rho \searrow \) on \( (a, b) \).

4. Determine the monotonicity patterns of the following ratios \( r(x) \):
(a) $\frac{\sin x}{x}$ on the interval $(0, \frac{\pi}{2})$;
(b) $\frac{(1+x)\ln(1+x)-x}{x^2}$ on the interval $(0, \infty)$;
(c) $\frac{1+e^x}{x}$ on the interval $(0, \infty)$.

5. Show that, in Example 4, the current concentration of the chemical in the pipe coincides with the current value of $\rho = f'/g'$.

6. Suppose that points $A$, $B$, and $C$ are on the same road, $B$ between $A$ and $C$. Cars 1 and 2 start simultaneously at points $A$ and $B$, respectively, towards point $C$. At time moments $t \geq 0$, the cars’ respective speeds are $v_1(t)$ and $v_2(t)$, and their respective distances from point $A$ are $x_1(t)$ and $x_2(t)$. Suppose that the speed ratio $v_1(t)/v_2(t)$ steadily increases in $t$. What can be said about the monotonicity pattern of the ratio $x_1(t)/x_2(t)$ of the distances?

2.2 Solutions to the exercises

Solution to Exercise 1. One has

$$r' = \left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2} = \left(\frac{f'}{g'} - \frac{f}{g}\right) \frac{g'}{g} = (\rho - r) \frac{g'}{g}. $$

Solution to Exercise 2. Rewrite identity (1) as

$$\rho - r = \frac{r'}{g'/g};$$

this and condition $g'/g > 0$ yields the result: (i) if $r'(u) = 0$ then $\rho(u) = r(u)$ and (ii) if $r' < 0$ on $(u, u + h)$ then $\rho - r > 0$ and hence $\rho > r$ on $(u, u + h)$.

Solution to Exercise 3. All the subcases of the special-case rule can be verified in the same manner. For instance, if $f(b) = g(b) = 0$ and $\rho \not\in \mathbb{R}$ on $(a, b)$, then the condition $g(b) = 0$ implies that $g$ and $g'$ have opposite signs on $(a, b)$, so that $g'/g < 0$ on $(a, b)$. In view of the conditions $f(b) = g(b) = 0$
and the mean-value theorem, for every $x$ in the interval $(a, b)$ there is some point $c$ in the interval $(x, b)$ such that

$$r(x) = \frac{f(x)}{g(x)} = \frac{f(b) - f(x)}{g(b) - g(x)} = \frac{f'(c)}{g'(c)} = \rho(c) < \rho(x),$$

the latter inequality taking place because $c > x$ and $\rho \downarrow$. Therefore, $\rho > r$ on $(a, b)$. Now it follows, in view of identity (1) and condition $g'/g < 0$, that $r' < 0$ and hence $r \searrow$ on $(a, b)$. This verifies the special-case rule in the subcase when $f(b) = g(b) = 0$ and $\rho \searrow$.

**Solution to Exercise 4.**

(a) This exercise is similar to Example 1. Letting here $f(x) = \sin x$ and $g(x) = x$, one has $f(0) = g(0) = 0$ and $\rho(x) = \cos x \searrow$ on $(0, \pi/2)$, whence, by the special-case rule, $r \searrow$ on $(0, \pi/2)$.

(b) This exercise is similar to Example 2. Letting here $f(x) = (1 + x)\ln(1 + x) - x$ and $g(x) = x^2$, one has $f(0) = g(0) = f'(0) = g'(0) = 0$ and $f''(x) = \frac{1}{2(1+x)} \searrow$ on $(0, \infty)$, whence, by the special-case rule, $r \searrow$ on $(0, \infty)$.

(c) This exercise is similar to Example 3. Letting here $f(x) = 1 + e^x$ and $g(x) = x$, one has $g'(x)/g(x) = 1/x > 0$ and $\rho(x) = e^x \nearrow$ on $(0, \infty)$, whence, by Table 1, $r \nearrow$ or $\nearrow$ or $\searrow \nearrow$ on $(0, \infty)$. Since $\lim_{x \to 0^+} r(x) = \infty$ and (by l’Hospital’s rule for limits) $\lim_{x \to \infty} r(x) = \lim_{x \to \infty} \rho(x) = \infty$, the answer is $\searrow \nearrow$. (Alternatively, instead of taking the limits, one could consider three trial points, say at $x$ equal 0.1, 1, and 10.)

**Solution to Exercise 5.** The change, $g(t+h) - g(t)$, of the total amount of solution in the tank over a time interval $(t, t+h)$ equals the total amount of solution that flows from the pipe into the tank during this time interval. The latter amount is $\approx kh$, if $k = k(t)$ is the solution inflow rate at time $t$ and the length $h$ of the time interval is small. Hence,

$$g(t+h) - g(t) \approx kh, \tag{3}$$

Let $c = c(t)$ be the concentration of the chemical in the pipe at time $t$. Then $-\frac{1}{k}$ of the total amount, $\approx kh$, of solution that flows from the pipe into the
tank over the time interval \((t, t + h)\) — the amount of the chemical will be
\(\approx ck h\). Hence, similarly to equation (3), one has
\[
f(t + h) - f(t) \approx ck h,
\]
so that
\[
c = \frac{ckh}{kh} \approx \frac{f(t + h) - f(t)}{g(t + h) - g(t)} = \frac{f(t + h) - f(t)}{h} \frac{f'(t)}{g'(t)} \to 0 \Rightarrow f'(t) = \rho(t).
\]
Thus indeed, the current concentration, \(c\), in the pipe coincides with the
current value of \(\rho\).

Solution to Exercise 6. By the first row of Table 1, the ratio \(x_1(t)/x_2(t)\)
can switch at most once, and only from \(\smaller\) to \(\smaller\). Moreover, initially, at times
\(t\) close to 0, the ratio \(x_1(t)/x_2(t)\) can only be increasing, from 0 to positive
values (because \(x_1(0) = 0\)). Hence, no switch can actually occur here, so
that the ratio \(x_1(t)/x_2(t)\) will be always increasing.

3 Supplementary and Advanced Material

3.1 Supplementary and Advanced Exercises

7. Here, for some differentiable functions \(f\) and \(g\) such that \(g\) and \(g'\) do not change sign on an
interval \((a, b)\), the graphs of the two ratios,
\(r = f/g\) and \(\rho = f'/g'\), on \((a, b)\) are shown.
Which is which? What is the sign of \(g'/g\) on
\((a, b)\)?

8. Using Table 1, one can generally determine the monotonicity pattern of \(r\)
given that of \(\rho\), however complicated the latter might be. This is illustrated
by Table 2. Verify the table.
Table 2: What if \( \rho \) itself is not monotonic?

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( g'/g )</th>
<th>( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \uparrow \downarrow )</td>
<td>( &gt; 0 )</td>
<td>( \uparrow ) or ( \downarrow ) or ( \uparrow ) or ( \downarrow ) or ( \uparrow \downarrow )</td>
</tr>
<tr>
<td>( \uparrow )</td>
<td>( &gt; 0 )</td>
<td>( \uparrow ) or ( \downarrow ) or ( \uparrow ) or ( \downarrow ) or ( \uparrow \downarrow )</td>
</tr>
<tr>
<td>( \downarrow )</td>
<td>( &lt; 0 )</td>
<td>( \uparrow ) or ( \downarrow ) or ( \uparrow ) or ( \downarrow ) or ( \uparrow \downarrow )</td>
</tr>
<tr>
<td>( \downarrow \uparrow )</td>
<td>( &lt; 0 )</td>
<td>( \uparrow ) or ( \downarrow ) or ( \uparrow ) or ( \downarrow ) or ( \uparrow \downarrow )</td>
</tr>
</tbody>
</table>

9. Here we shall consider another proof of the the rules of Table 1 (as well as of the special-case rule), assuming additionally that \( \rho = f'/g' \) is differentiable (which will be the case if \( f \) and \( g \) are twice differentiable). Proofs without this extra assumption are given in [14, 19].

(a) Show that

\[
\text{sign}(r') = \text{sign} \tilde{\rho}, \quad \text{where} \quad \tilde{\rho} := (\rho g - f) \text{sign}(g').
\]

The sign function is defined by the formulas \( \text{sign} x = 1 \) if \( x > 0 \) and \( \text{sign} x = -1 \) if \( x < 0 \). The symbol \( \tilde{\rho} \) can be read “tilde-rho” or “rho-wiggle”.

(b) Show that

\[
\tilde{\rho}' = \rho' |g| \text{sign}(g'/g),
\]

which yields

\[
\text{sign}(\tilde{\rho}') = \text{sign}(\rho') \text{sign}(g'/g),
\]

so that, if \( \rho \) is monotonic on \((a, b)\), then \( \tilde{\rho} \) is so.

(c) But if \( \tilde{\rho} \) is monotonic, how many times (at most) can it switch sign on \((a, b)\)? What does this imply for the sign pattern of \( r'' \)? (Hint: recall equation (4).) Show now that the rules given in Table 3 hold. Note that the value \( \tilde{\rho}(a) \) exists at least in the sense of the limit of \( \tilde{\rho}(x) \) as \( x \downarrow a \); this limit exists (since \( \tilde{\rho} \) is monotonic) but may be infinite; similarly, for \( \tilde{\rho}(b) \).
Table 3: General rules for monotonicity, restated when $\rho$ is monotonic.

<table>
<thead>
<tr>
<th>$\tilde{\rho}(a)$</th>
<th>$\tilde{\rho}(b)$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\geq 0$</td>
<td>$\geq 0$</td>
<td>$\nearrow$</td>
</tr>
<tr>
<td>$&gt; 0$</td>
<td>$&lt; 0$</td>
<td>$\searrow$</td>
</tr>
<tr>
<td>$&lt; 0$</td>
<td>$&gt; 0$</td>
<td>$\nearrow$</td>
</tr>
<tr>
<td>$\leq 0$</td>
<td>$\leq 0$</td>
<td>$\searrow$</td>
</tr>
</tbody>
</table>

(d) Deduce the rules of Table 1 from those of Table 3.

(e) Deduce the special-case rule for monotonicity from those of Table 3.

Assume for simplicity that $g'$ is nonzero at the endpoints $a$ and $b$, so that the values $\rho(a)$ and $\rho(b)$ are well defined and finite. (Hint: Show that conditions $f(a) = g(a) = 0$ imply that $g'/g > 0$ on $(a, b)$ and $\tilde{\rho}(a) = 0$; deal similarly with $b$ in place of $a$.)

The rules of Table 3 are very efficient and provide us with a definite monotonicity pattern of $r$. We see that the function $\tilde{\rho}$ is key to understanding why the rules for monotonicity work. To use the insight that we gain from Exercise 9, let us consider the following more general version of Example 3.

10. Determine the monotonicity pattern of ratio

$$
\tilde{r}(x) = \frac{-c + \sin x}{x}
$$

on $(0, \frac{\pi}{2})$ as it depends on the parameter $c$, which may take on any real values.

11. Use rules for monotonicity to determine the convexity pattern of the ratio considered in Example 2:

$$
\tilde{r}(x) = \frac{e^x - 1 - x}{x^2}
$$

for $x \neq 0$ and $r(0) = \frac{1}{2}$.

12. Determine the monotonicity pattern of the ratio

$$
\tilde{r}(x) = \frac{x^a e^{-x}}{\int_x^\infty u^a e^{-u} du}, \quad x > 0,
$$

as it depends on the real-valued parameter $a$. 

16
13. [21] Show that equation

\[ \ln \frac{1 - u}{-\ln u} - 1 - \frac{1}{2} \left( \frac{1 + u}{1 - u} \right) \ln u = 0 \] (8)

in \( u \in (0, 1) \) has exactly one solution.

One can find more such examples, as well as ideas for more examples, in cited literature. Clearly, the stated rules for monotonicity could be helpful when \( f' \) or \( g' \) can be expressed simpler than \( f \) or \( g \), respectively. Such functions \( f \) and \( g \) are essentially the same as the functions that could be taken to play the role of \( u \) in the integration-by-parts formula \( \int u dv = uv - \int v du \); this class of functions includes polynomial, logarithmic, and inverse trigonometric functions, as well as non-elementary “anti-derivative” functions of the form \( x \mapsto \int_a^x h(u) \, du \) or \( x \mapsto \int_x^b h(u) \, du \).

Here is another “modeling” exercise.

14. Imagine that a bird flies off the top of a tall thin post and continues flying so that its projection onto the ground moves along a straight line through the bottom of the post steadily away from it, while at the bottom of the post an observer remains standing. The bird is flying steeper and steeper up; that is, the bird’s trajectory is convex. What can be said about the pattern of the bird’s flight, in terms of the bird appearing rising and/or descending to the observer? Assume that the bird is infinitely small and the observer watches its flight with one eye only, so that he does not perceive any change in the distance of the bird from him. Assume also that the earth is flat!

One can develop many similar “modeling” exercises, say for such ratios as Debt-to-Equity, Predators-to-Prey, Students-to-Faculty, and Current Year’s Sales-to-Last Year’s Sales. Especially for the latter ratio, a discrete-time setting should be more appropriate; cf. Corollaries 1 and 2 in [20].

15. Show that l’Hospital’s rule for limits of the form \( \frac{\infty}{\infty} \) does not have a “special-case” analogue for monotonicity. (Hint: You may review the “bird” problem, Problem 14. Other examples demonstrating the same thesis were
given in [5]. Of course, in all such examples, while the special-case rule for monotonicity does not work, the general rules still do.

3.2 Solutions to the supplementary and advanced exercises

Solution to Exercise 7. If we had here \( r > \rho \) on \((a, b)\) then, by identity (1), \( r' \) would not change sign on \((a, b)\), so that \( r \) would be monotonic on \((a, b)\). However, here the greater of the two functions appears to have the \( \nearrow \searrow \) pattern, and then it is non-monotonic. Hence, the greater function must be \( \rho \), so that \( \rho > r \) on \((a, b)\). Moreover, it then appears that the smaller function, \( r \), has a positive derivative, \( r' \), on \((a, b)\). Hence, by identity (1), \( g'/g > 0 \) on \((a, b)\).

Solution to Exercise 8. Let us verify row 1 of Table 2 (the other rows are done similarly). In view of Table 1 and given that \( g'/g > 0 \) and \( \rho \) switches from \( \nearrow \) to \( \searrow \) on \((a, b)\), we see that \( r \) switches from one of the patterns \( \nearrow \) or \( \searrow \) to one of the patterns \( \nearrow \) or \( \nearrow \searrow \). Concatenating these patterns, one has one of the patterns \( \nearrow \) or \( \searrow \) or \( \nearrow \searrow \) or \( \nearrow \searrow \searrow \searrow \) (for instance, concatenating patterns \( \nearrow \searrow \) and \( \nearrow \) results in pattern \( \nearrow \searrow \)).

Solution to Exercise 9. (a) Using the rule for the derivative of a ratio and some simple algebra, one gets 
\[
\frac{\mid g' \mid}{g'} \rho \text{'},
\]
whence the relation between the signs.

(b) Note that \( \text{sign}(g') \) in the right-hand side of equation (5) is a constant factor (equal \( \pm 1 \)), because we assume that \( g' \) does not change sign on the interval \((a, b)\). Now differentiate both sides of equation (5) to get
\[
\rho' = ((\rho g - f) \text{sign}(g'))' = (\rho g - f)' \text{sign}(g') = (\rho' g + \rho g' - f') \text{sign}(g') = \rho' g \text{sign}(g') = \rho' |g| \text{sign}(g'/g).
\]
(c) If \( \dot{\rho} \) is monotonic, it can switch its sign at most once on \((a, b)\). Now, by equation (4), we see that \( r' \) can switch its sign at most once on \((a, b)\), so that \( r \) can switch at most once from \( \nearrow \) to \( \searrow \), or vice versa, on \((a, b)\). To verify Table 3 assume, for instance, that (as in the 2nd row of Table 3), \( \dot{\rho}(a) > 0 \) and \( \dot{\rho}(b) < 0 \). Then (being monotonic) \( \dot{\rho} \) switches sign from + to − on \((a, b)\) and hence so does \( r' \), which implies that \( r \) switches from \( \nearrow \) to \( \searrow \); that is, \( r \nearrow \searrow \) on \((a, b)\). Similarly one can verify rows 1, 3, and 4 of Table 3.

(d) For instance, if (as in the 1st row of Table 1) \( \rho \nearrow \) and \( g'/g > 0 \) then, by (7), \( \dot{\rho} \nearrow \); therefore, \( \dot{\rho} \) may switch sign on \((a, b)\) only from − to +. If it does so then, by Table 3, \( r \searrow \nearrow \); otherwise, \( r \nearrow \) or \( \searrow \) on \((a, b)\). This verifies the 1st row of Table 1; its other three rows are verified quite similarly.

(e) For instance, if \( f(a) = g(a) = 0 \) then the condition \( g(a) = 0 \) implies that \( \text{sign} g = \text{sign}(g') \) on \((a, b)\), so that \( g'/g > 0 \) on \((a, b)\). If, for instance, we also have \( \rho \nearrow \) on \((a, b)\), then equation (7) implies that \( \dot{\rho} \nearrow \) on \((a, b)\). But, in view of equation (5) and conditions \( f(a) = g(a) = 0 \), one has

\[
\dot{\rho}(a) = (\rho(a)g(a) - f(a))\text{sign}(g'(a)) = 0.
\]

This, together with \( \dot{\rho} \nearrow \) on \((a, b)\), implies that \( \dot{\rho}(b) > 0 \). It follows by the 1st row of Table 3 that \( r \nearrow \) on \((a, b)\). This verifies the special-case rule in the subcase when \( f(a) = g(a) = 0 \) and \( \rho \nearrow \) on \((a, b)\). The other three subcases (when \( f(b) = g(b) = 0 \) or \( \rho \searrow \) on \((a, b)\)) are verified similarly.

Solution to Exercise 10. Let \( f(x) := -c + \sin x \) and \( g(x) := x \), so that \( r = f/g \). Note that here

\[
\rho(x) = \frac{f'(x)}{g'(x)} = \frac{\cos x}{1} = \cos x \searrow
\]

on \((0, \frac{\pi}{2})\), the same as in the special case of \( c = \frac{1}{2} \), considered in Example 3 (so that \( \rho \) does not depend on the choice of \( c \)). By equation (5),

\[
\dot{\rho}(x) = (\rho(x)g(x) - f(x))\text{sign}(g'(x)) = x \cos x + c - \sin x,
\]

whence

\[
\dot{\rho}(0) = c \quad \text{and} \quad \dot{\rho}(\frac{\pi}{2}) = c - 1.
\]
Now we conclude by Table 3:

- if \( c \geq 1 \), then \( \bar{\rho}(0) = c \geq 0 \) and \( \bar{\rho}\left(\frac{x}{2}\right) = c - 1 \geq 0 \), so that \( r \nearrow \);
- if \( 0 < c < 1 \), then \( \bar{\rho}(0) = c > 0 \) and \( \bar{\rho}\left(\frac{x}{2}\right) = c - 1 < 0 \), so that \( r \nwarrow \);
- if \( c \leq 0 \), then \( \bar{\rho}(0) = c \leq 0 \) and \( \bar{\rho}\left(\frac{x}{2}\right) = c - 1 \leq 0 \), so that \( r \searrow \).

In particular, we confirm what we established in Example 3: that for \( c = \frac{1}{2} \), one has \( r \searrow \).

In part (a) of this picture, we can see that for all values of parameter \( c \), the monotonicity pattern of \( \bar{\rho} \) is the same as that of \( \rho \), just as follows by equation (7). Also, \( \bar{\rho} \) switches sign (only from + to −) only when \( 0 < c < 1 \), and in this case part (b) of the picture shows that \( r \nearrow \) on \((0, \frac{\pi}{2})\). If \( c \geq 1 \), then \( \bar{\rho} \) remains positive and hence \( r \searrow \) on the entire interval \((0, \frac{\pi}{2})\). If \( c \leq 0 \), then \( \bar{\rho} \) is negative and hence \( r \searrow \) on \((0, \frac{\pi}{2})\).

\textbf{Solution to Exercise 11.} Consider

\[ r'(x) = R(x) := \frac{F(x)}{G(x)} \text{ for } x \neq 0, \text{ where } F(x) := e^x(x-2)+x+2 \text{ and } G(x) := x^3. \]
Then \( F(0) = G(0) = 0, \ F'(0) = G'(0) = 0 \), and (for \( x \neq 0 \))
\[
\frac{F''(x)}{G''(x)} = \frac{e^x}{6}.
\]
Hence, \( r' = Rf / g \) on \((0, \infty)\) and on \((-\infty, 0)\), so that \( r \) is convex on \((-\infty, \infty)\).

**Solution to Exercise 12.** Let \( f(x) := x^a e^{-x} \) and \( g(x) := \int_x^\infty u^a e^{-u} \) for \( x > 0 \), so that \( r = f/g \). Then \( f(\infty) = g(\infty) = 0 \), and \( \rho(x) = f'(x)/g'(x) = 1 - a/x \), so that \( \rho \) is increasing or decreasing on \((0, \infty)\) according to whether \( a > 0 \) or \( a < 0 \). Hence, by the special-case rule, \( r \) is increasing or decreasing on \((0, \infty)\) according to whether \( a > 0 \) or \( a < 0 \). (If \( a = 0 \) then \( r = 1 \) on \((0, \infty)\).) One can also note that, by l'Hospital's rule, \( r(\infty) = \rho(\infty) = 1 \).

It follows that \( r < 1 \) or \( r > 1 \) on \((0, \infty)\) according to whether \( a > 0 \) or \( a < 0 \).

**Solution to Exercise 13.** Let
\[
h(u) := \ln \frac{1 - u}{-\ln u} - 1 - \frac{1}{2} \frac{(1 + u)\ln u}{1 - u}, \tag{9}
\]
the left-hand side of (8). Here and in the rest of this solution, it is assumed that \( 0 < u < 1 \), unless specified otherwise. Then
\[
h'(u) = r_1(u) := \frac{f_1(u)}{g_1(u)}, \tag{10}
\]
where \( f_1(u) := 2 \ln^2 u + (\frac{1}{u} + 2 - 3u) \ln u + 2(u + \frac{1}{u}) - 4 \) and \( g_1(u) := -2(1 - u)^2 \ln u \). Let next
\[
r_2(u) := \frac{f_1'(u)}{g_1'(u)} = \frac{f_2(u)}{g_2(u)}, \tag{11}
\]
where \( f_2(u) := (\frac{3}{u} - \frac{1}{u^2}) \ln u + \frac{1}{u} - \frac{1}{u^2} \) and \( g_2(u) := 4 \ln u - \frac{2}{u} + 2 \), and then
\[
r_3(u) := \frac{f_2'(u)}{g_2'(u)} = \frac{f_3(u)}{g_3(u)}, \tag{12}
\]
where \( f_3(u) := (2 - 3u) \ln u + 1 + 2u \) and \( g_3(u) := 2u(1 + 2u) \).

One has \( \frac{f''(u)}{g''(u)} = -\frac{1}{8} (\frac{3}{u} + \frac{3}{u^2}) \), which is increasing; moreover, \( \frac{d}{du} \frac{f_1'(u)}{g_1(u)} \) tends to \(-\infty < 0 \) and \(-29/50 < 0 \) as \( u \downarrow 0 \) and \( u \uparrow 1 \), respectively. Hence, by Table 1, \( \frac{f_1(u)}{g_1(u)} \) is decreasing (in \( u \in (0, 1) \)).
Next, by (12), \( r'_3(0+) = \infty > 0 \) and \( r'_3(1-) = -2/3 < 0 \). Hence, by Table 1, \( r_3 \searrow \) (on \( 0, 1 \))

By (11), \( r'_2(0+) = \infty > 0 \) and \( f_2(1) = g_2(1) = 0 \). Hence, by Table 1 and the special-case rule for monotonicity, \( r_2 \searrow \) (on \( 0, 1 \)).

By (10), \( r'_1(0+) = \infty > 0 \) and \( r'_1(1-) = -1/4 < 0 \). Hence, by Table 1, \( h' = r_1 \searrow \) (on \( 0, 1 \)). Moreover, \( h'(0+) = -\infty < 0 \) and \( h'(1-) = \frac{1}{2} > 0 \). Hence, for some \( \beta \in (0, 1) \), one has \( h' < 0 \) on \( (0, \beta) \) and \( h' > 0 \) on \( (\beta, 1) \).

Hence, \( h \searrow \) on \( 0, 1 \). Moreover, \( h(0+) = \infty \) and \( h(1-) = 0 \). It follows that the equation \( h(u) = 0 \) has a unique root \( u \) in \( 0, 1 \).

**Solution to Exercise 14.** Let \( f \) and \( g \) be, respectively, the vertical and horizontal components of the vector from the observer to the bird. At any given time moment \( t \), the bird will appear rising to the observer if and only if the angle \( \theta := \arctan r \) (or, equivalently, its tangent, \( r = f/g \)) is increasing at the time \( t \). Because the slope coefficient \( \rho = f'/g' \) of the bird’s trajectory is increasing, the pattern can switch at most once in time, and only from the descending mode to the ascending one (by the first line of Table 1). Moreover, initially \( r \) can only be decreasing, from \( \infty \) to finite values.

Thus, the bird will either (i) appear descending all the time or (ii) there will be exactly one switch – from the descending mode to the ascending one. In particular, such a switch will necessarily occur if the slope coefficient \( \rho = f'/g' \) tends to infinity; this can be seen from (1). (Indeed, if a switch never occurs, then \( r \) must be decreasing all the time, to some finite value. Hence and because \( \rho \) tends to \( \infty \), eventually one will have \( \rho > r \), and then (1) will imply that eventually \( r' > 0 \) and hence \( r \) is increasing, which is a contradiction.) The pictures on the left illustrate the two possibilities.
This example not only helps visualize the rules for monotonicity but also shows that, without loss of generality, one may assume that \( g(x) = x \) for all \( x \) (as actually was the case in Example 3 and almost so in Example 1); then \( \rho = f' \), and the condition that \( \rho \) is nondecreasing (say) will simply mean that \( f \) is convex. Analytically, this reduction can be achieved by introducing the “new variables” \( X := g(x) \) and \( Y := f(x) \), which will yield a “parametric” definition of a function \( Y = h(X) \), where \( h := f \circ g^{-1} \), so that \( r(x) = \frac{h(X)}{X} \). While such a reduction makes the rules more palpable, it does not seem to lead to a significant overall simplification of the proofs presented here.

**Solution to Exercise 15.** The “bird” problem suggests that one can obtain the required effect by taking \( g(x) = x \) and a convex function \( f \) such that \( f(x) > x \) for all \( x > 0 \) and \( f(x) \) is asymptotic to \( x \) as \( x \to \infty \); that is, take \( f(x) = x + f_1(x) \), where \( f_1 \) is positive and convex on \((0, \infty)\) and \( f_1(x) \to 0 \) as \( x \to \infty \). For example, take \( f(x) = x + e^{-x} \). Then both \( f(x) \) and \( g(x) \) tend to \( \infty \) as \( x \to \infty \), and \( \rho(x) = 1 - e^{-x} / x \). Yet, \( r(x) = 1 + \frac{1}{xe^x} \) on \((0, \infty)\).

**References**


