The effect of heat conduction in the vapor on the dynamics of downflowing condensate

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A vapor fills the gap between two vertical plates, one hot and one cold. The temperatures are adjusted so that condensate forms on the cold wall. It is the dynamics of the system that is examined. The paper extends the one-sided model of evaporation–condensation to account the heat conduction in the vapor phase, which turns out to be important in many condensation problems. For the considered flow, both vapor recoil and Marangoni effect are stabilizing; as a result, the condensate becomes unstable at nonzero Reynolds numbers in contrast to the usual film flow down a vertical wall. A nonlinear evolution equation is derived and analyzed for the interaction of viscous shear and evaporation–condensation. It turns out that the one-sided model of heat and mass transfer gives a very good description of the initial stage of thin-film growth; in later stages, however, the heat conduction through the vapor becomes important when the film is sufficiently thick. © *2002 American Institute of Physics.* [DOI: 10.1063/1.1425845]

I. INTRODUCTION

We consider cold and hot vertical walls which encloses a vapor. The vapor condenses on the cold wall forming a thin liquid film that flows down under gravity, Fig. 1. This system is related to several others: Downflowing homogenous film without temperature gradients, evaporation–condensation of the film lying on the horizontal plate, and two-phase flow in the channel without mass transfer. The flows of condensed film are encountered in many heat-transfer systems.

In the context of the liquid films, evaporation–condensation was studied by Burelbach, Bankoff, and Davis.[1] They proposed the one-sided model of heat and mass transfer in the liquid films which allows one to decouple and study the dynamics of the liquid separately from that of the vapor. This approach was used[2] to study the dynamics of heated downflowing film. Dynamics of the vapor film boiling regime was studied by Panzarella, Davis, and Bankoff.[3] They derived a long-wavelength evolution equation for the thickness of the film, and considered the influence of various physical effects on the film dynamics.

A goal of the present work is to contrast the one-sided model of heat transfer[1] with an extension that accounts for the heat transfer in the vapor. The isothermal vertically falling film is always unstable, i.e., its critical Reynolds number R_c is zero. For the condensed film flow the situation is different. Here both vapor recoil and Marangoni effects are stabilizing and hence the falling condensate becomes unstable at nonzero Reynolds numbers. We derive the long-wavelength nonlinear evolution equation of the condensed film surface, and analyze the interaction of viscous shear and evaporation–condensation.

This work on a falling film extends the idea of VanHook *et al.*,[4] who showed for long-scale thermocapillary convection in a horizontal layer, that if the film is thick, then the quantitative comparison with experiments has to account for heat transport in the gas. We believe, therefore, that incorporation of the outer-wall influence in two-phase heat- and mass-transfer problems will improve the models of these complex systems.

II. STATEMENT OF THE PROBLEM

Consider two parallel, vertical, infinite walls having separation d as shown in Fig. 1. The temperature of the cold wall T_C is below the saturation temperature T_S and the temperature of the hot wall T_H is above T_S. The liquid with density ρ and dynamic velocity μ condenses on the cold wall and constitutes a thin film flowing down under the action of gravity $\boldsymbol{g}$. The pressure in the vapor is taken to be p_0, a constant which can be achieved by making the hot wall porous so that pressure variations can be suppressed. There exists a steady downflowing film with its interface temperature equal to T_S. We shall investigate the two-dimensional nonlinear stability of the film. The vapor–liquid surface $\underline{x} = \underline{h}(z,t)$ is a function of lateral coordinate z and time t.

Burelbach *et al.*[1] proposed the one-sided model of evaporation–condensation of the liquid films. The idea of the approach is to consider the limit when density, dynamic viscosity, and thermal conductivity of the vapor are much less than those in the liquid but the mass flux across the interface

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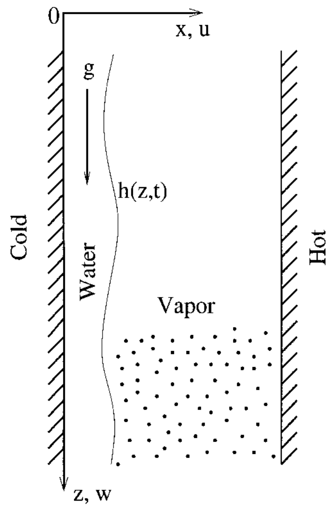

FIG. 1. Geometry of the heat transfer system.

continues to hold. This method allows one to decouple the dynamics of the liquid from that of the vapor. We also shall disregard the hydrodynamical motion of the vapor, but shall retain the effects of heat transfer in the vapor, extending the one-sided model to allow two-phase heat conduction. The vapor in the film system with nonuniform temperature field easily conducts the heat from the outer wall to the surface of the liquid and affects the evaporation–condensation. This influence of the outer wall through the vapor phase was found to be important in other heat-transfer problems,[4] and is absent in the one-sided model of mass transfer.

Burelbach *et al.*[1] gave a detailed discussion of the pertinent equations, boundary conditions, underlying physical assumptions and involved dimensionless parameters of the evaporation–condensation process. We, therefore, shall summarize the statement of the problem, omitting detailed discussions.

The motion and the energy balance of the liquid and vapor are governed by the following set of equations:

$$\nabla\cdot\boldsymbol{v}=0,\quad \rho(\boldsymbol{v}_t+\boldsymbol{v}\cdot\nabla\boldsymbol{v})=-\nabla p+\rho\boldsymbol{g}+\mu\nabla^2\boldsymbol{v},\tag{1}$$

$$T_t+\boldsymbol{v}\cdot\nabla T=\kappa\nabla^2 T,\quad \nabla^2 T^{(V)}=0.\tag{2}$$

Here z,x are streamwise and normal coordinates, respectively; $\boldsymbol{v}=(w,u)$ are the corresponding velocity components of the liquid; t is the time; p is the pressure in the liquid T is the liquid temperature, $T^{(V)}$ is the vapor temperature, κ is the thermal diffusivity of the liquid.

At the walls, the temperature is given, and there is no slip of liquid

$$T(0)=T_C,\quad T^{(V)}(d)=T_H,\quad u(0)=w(0)=0.\tag{3}$$

In addition there are conditions on the interface $x=h(z,t)$. Liquid is supposed to evaporate in the direction normal to the interface; therefore, the normal components of the velocity field in both phases are related through the mass flux of the evaporation–condensation:

$$J=\rho(\boldsymbol{v}-\boldsymbol{v}^{(I)})\cdot\boldsymbol{n}=\rho^{(V)}(\boldsymbol{v}^{(V)}-\boldsymbol{v}^{(I)})\cdot\boldsymbol{n},\qquad \boldsymbol{n}=\frac{(-h_z,1)}{(1+h_z^2)^{1/2}}.\tag{4}$$

Here $\boldsymbol{v}^{(V)}$ is the vapor velocity, $\boldsymbol{v}^{(I)}$ is the velocity of the (nonmaterial) interface, $\rho^{(V)}$ is the vapor density, assumed to be constant, and J is the mass flux due to vaporization.

In the energy relation on the interface, we disregard the viscous dissipation, and kinetic energy of both the phases. As was shown by Burelbach *et al.*, these quantities are very small compared to the heat of vaporization, and the conductive heat fluxes in both phases. As a result, the energy balance on the interface reads

$$k\nabla T\cdot\boldsymbol{n}-k^{(V)}\nabla T^{(V)}\cdot\boldsymbol{n}=JL.\tag{5}$$

Here L is the latent heat of vaporization, and $k^{(V)}$ is the thermal conductivity of the vapor.

The normal-stress condition balances the jump in normal stress with the surface tension and vapor recoil

$$J(\boldsymbol{v}^{(V)}-\boldsymbol{v}^{(I)})\cdot\boldsymbol{n}-\boldsymbol{\mathcal{T}}\cdot\boldsymbol{n}\cdot\boldsymbol{n}-p_0+\sigma(T)\nabla\cdot\boldsymbol{n}=0.\tag{6}$$

Here $\boldsymbol{\mathcal{T}}=-p\boldsymbol{I}+2\mu\boldsymbol{S}$ is the stress tensor of the liquid with $\boldsymbol{S}$ being the deformation rate tensor, $\sigma(T)$ is the surface tension, and $-\nabla\cdot\boldsymbol{n}$ is the mean curvature of the interface $h(z,t)$.

The shear-stress boundary condition balances the jump in shear with the surface-tension gradient caused by temperature variations (the Marangoni effect[5])

$$J(\boldsymbol{v}^{(V)}-\boldsymbol{v}^{(I)})\cdot\boldsymbol{t}-\boldsymbol{\mathcal{T}}\cdot\boldsymbol{n}\cdot\boldsymbol{t}=-\nabla\sigma\cdot\boldsymbol{t}.\tag{7}$$

The linear equation of state relates the surface tension σ to the temperature

$$\sigma(T)=\sigma_0-\gamma(T-T_S).\tag{8}$$

Here σ_0 is the surface tension at the saturation temperature T_S; usually $\gamma>0$.

Temperatures of the liquid T and vapor $T^{(V)}$ on the interface are equal

$$T=T^{(V)}.\tag{9}$$

The last condition on the free surface is the constitutive equation, relating mass flux J at the interface to the local surface temperature. We use the linearized expression, derived from the kinetic theory:[1,6]

$$J=\frac{\alpha\rho^{(V)}L}{T_S^{3/2}}\left(\frac{M_w}{2\pi R_g}\right)^{1/2}(T-T_S).\tag{10}$$

Here M_w is the molecular weight, R_g is the universal gas constant, and α is the accommodation coefficient.

In addition to that, the pertinent initial data should be specified for the temperature, velocity field, and film thickness.

III. EXTENSION OF ONE-SIDED MODEL AND ITS SCALINGS

To have an idea about typical scales of the problem, consider the simplest basic steady unidirectional film flow with thickness h_0. The temperature on the film surface $T(h_0)$

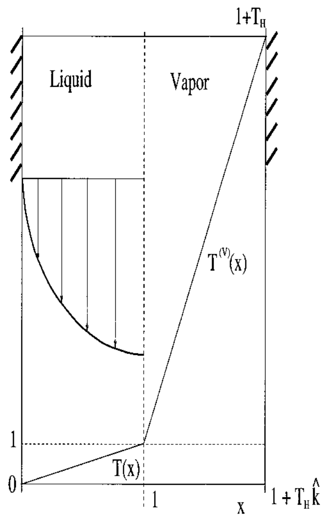

FIG. 2. Distribution of temperature and velocity of downflowing condensate.

is equal to the saturation temperature T_S. The solution of Eqs. (1)–(10) for the temperature distribution and thickness h_0 of such basic flow are

$$\frac{h_0}{d}=\frac{k(T_S-T_C)}{\Theta},\quad \Theta=k(T_S-T_C)+k^{(V)}(T_H-T_S), \tag{11}$$

$$T(x)=T_C+\frac{\Theta}{d}x,\quad T^{(V)}(x)=T_H+\frac{\Theta}{k^{(V)}d}x. \tag{12}$$

We now nondimensionalize the governing system of equations and boundary conditions. The temperature of the cold wall T_C is taken as the reference, and temperature difference $T-T_C$ is scaled on the temperature difference $\Delta T=T_S-T_C$. Length is scaled to the equilibrium thickness h_0 from (11), and T_H is the dimensionless excess of the hot wall temperature over the saturation temperature. Velocity is scaled on $\overline{U}=gh_0^2\rho/\mu$, where g is the gravity acceleration; pressure is referred to $\rho g h_0$; time is in units of $h_0/\overline{U}$. Mass flux is scaled on $k\Delta T/h_0 L$. (See Fig. 2.)

We consider a limiting case assuming that density and dynamic viscosity of the vapor are much smaller than that in the liquid. Formally, we take the limit similar to that applied by Burelbach *et al.*[1]

$$\frac{\rho^{(V)}}{\rho}\to 0,\quad \frac{\mu^{(V)}}{\mu}\to 0, \tag{13}$$

but do not take the limit $k^{(V)}/k\to 0$. However, the product of vapor density (which is small) and vapor velocity (which might be large) is retained in the mass-balance condition on the interface. This is similar to the Boussinesq approach in convection where density variations are neglected *except* in association with gravity. Not that the formal limit $k^{(V)}/k\to 0$ applied to the basic steady-state (11) and (12) results in an infinite temperature gradient in the vapor.

The equations of motion and energy balance now are

$$w_z+u_x=0, \tag{14}$$

$$R(w_t+ww_z+uw_x)=1-p_z+w_{zz}+w_{xx}, \tag{15}$$

$$R(u_t+wu_z+uu_x)=-p_x+u_{zz}+u_{xx}, \tag{16}$$

$$RP(T_t+wT_z+uT_x)=T_{zz}+T_{zz},\quad T_{zz}^{(V)}+T_{xx}^{(V)}=0. \tag{17}$$

Here

$$R=\frac{\overline{U}h_0\rho}{\mu}=\frac{gh_0^3\rho^2}{\mu^2},\quad P=\frac{\nu}{\kappa}, \tag{18}$$

are the Reynolds number of the liquid film, and the Prandtl number, respectively.

Boundary conditions at the walls are

$$T(0)=0,\quad T^{(V)}(1+\hat{k}T_H)=T_H+1,$$
$$w(0)=u(0)=0. \tag{19}$$

Here $1+\hat{k}T_H$ is the dimensionless position of the hot wall, and $\hat{k}=k^{(V)}/k$ is the ratio of thermal conductivities.

After scaling, taking the limit (13), and using the continuity equation (14), boundary conditions of the film surface $h(z,t)$ become

energy balance:

$$(T_x-T_zh_z)-\hat{k}(T_x^{(V)}-T_z^{(V)}h_z)=J(1+h_z^2)^{1/2}, \tag{20}$$

normal stress:

$$\frac{E^2}{R\hat{\rho}}J^2-p+p_0+2\frac{u_x(h_z^2-1)-h_z(u_z+w_x)}{1+h_z^2}=S\frac{h_{zz}}{(1+h_z^2)^{3/2}}, \tag{21}$$

shear stress:

$$(1-h_z^2)(u_z+w_x)-4h_zu_x=-SC(T_z+T_xh_z)(1+h_z^2)^{1/2}, \tag{22}$$

constitutive equation:

$$KJ=T-1, \tag{23}$$

mass balance:

$$\frac{E}{R}J=(\boldsymbol{v}-\boldsymbol{v}^{(I)})\cdot\boldsymbol{n}, \tag{24}$$

kinematic condition:

$$h_twh_z-u=-\frac{E}{R}J(1+h_z^2)^{1/2}. \tag{25}$$

Here the dimensionless parameters

$$E=\frac{\kappa\Delta T}{L\mu},\quad \hat{\rho}=\frac{\rho^{(V)}}{\rho},\quad S=\frac{\sigma_0}{\rho g h_0^2},\quad C=\frac{\Delta T\lambda}{\sigma_0},$$
$$K=\frac{kT_S^{3/2}}{\alpha h_0\rho^{(V)}L^2}\left(\frac{2\pi R_g}{M_w}\right)^{1/2}, \tag{26}$$

are the evaporation number, the density ratio, the nondimensionless surface tension, the capillary number, and the nonequilibrium parameter. The right-hand term in (20) measures the energy flux due to evaporation, the left-hand term in (21) measures the impact of the vapor recoil on the normal stress, the right-hand term in (22) governs the Marangoni effect, and the right-hand terns in (24) and (25) correspond to the evaporative part of the mass balance. The parameter K measures the degree of nonequilibrium at the evaporating surface; $K=0$ corresponds to the quasi-equilibrium limit, where the interface temperature is equal to the saturation value, $T=1$. $K^{-1}=0$ corresponds to the nonvolatile case, and the absence of mass transfer between phases.

We shall assume that surface tension is large enough to ensure that the resulting waves are long: $S\gg 1$. At the same time, $C\ll 1$, so that $SC=O(1)$. As a result, the effect of variable surface tension can be dropped from the normal-stress balance (21), but the Marangoni effect is retained in shear-stress condition (22).

IV. BASIC AND QUASI-STEADY FLOWS: ONE-SIDED MODEL VS TWO-SIDED

The basic state is the steady unidirectional flow of the film down a cold wall, without mass transfer between the liquid and vapor phases. Below, we shall find the quasi-steady quasi-one-dimensional flow of the film, i.e., varying slowly along the lateral coordinate z and in time.

The equations and boundary conditions governing the quasi-steady film flow are

$$p_x=0, \quad w_{xx}=-1, \quad T_{xx}=0, \quad T_{xx}^{(V)}=0, \tag{27}$$

$$\text{on } x=h(z,t): \quad -\frac{E^2}{R\hat{\rho}}J^2+p-p_0=-Sh_{zz},$$

$$T=T^{(V)}, \quad T_x=\hat{k}T_x^{(V)}+J, \tag{28}$$

$$KJ=T-1, \quad w_x=0, \quad h_t+\frac{\partial}{\partial z}\int_0^h w\,dx=-\frac{E}{R}J, \tag{29}$$

$$\text{on the wall} \quad x=0: w(0)=0, \quad T(0)=0,$$

$$T^{(V)}(1+\hat{k}T_H)=1+T_H. \tag{30}$$

The solution of the thermal problem gives

$$T(x)=\frac{h-1-\hat{k}\Omega}{\Sigma}x,$$

$$T^{(V)}(x)=1+T_H+\frac{\Omega}{\Sigma}(1+\hat{k}T_H-x), \tag{31}$$

$$J=\frac{1+\hat{k}T_H}{\Sigma}(1-h),$$

where

$$\Sigma(h)=h^2-K(1-h)-h(1+\hat{k}T_H)-\hat{k}K(h+T_H),$$

$$\Omega=K+T_HKT_H. \tag{32}$$

The temperature profile remains linear, with coefficients depending on the film thickness h. The pressure in the liquid is

$$p=p_0+\frac{E^2}{R\hat{\rho}}J^2-Sh_{zz}. \tag{33}$$

The second and third terms in (33) correspond to the influence of vapor recoil, and surface tension, respectively. The velocity field has the parabolic profile

$$w(x)=hx-\tfrac{1}{2}x^2. \tag{34}$$

Substitution of expressions (31) and (34) into the kinematic boundary condition (29) results in equation describing, in the first approximation, the evolution of the interface due to nonlinear effects, and mass exchange between liquid and vapor phases

$$h_t+h^2h_z=-\frac{E}{R}J. \tag{35}$$

When there is fully developed flow (independent of coordinate z, (35) becomes

$$\frac{dh}{dt}=-\frac{E}{R}J, \tag{36}$$

which describes the change in time of the thickness of the one-dimensional liquid–vapor system at constant vapor pressure. If $\hat{k}=0$, i.e., the heat conduction in the vapor is omitted, Eqs. (36) and (31) reduce to those of the one-sided model

$$\frac{dh}{dt}=\frac{E}{R}\frac{1}{h+K}, \quad T(x)=\frac{1}{h+K}x, \quad J=\frac{1}{h+K}. \tag{37}$$

Solution of Eq. (37) coincides with basic state of condensing film, found by Burelbach *et al.*[1] (up to rescaling of time).

Note that in the one-sided model all the heat released during the condensation is conducted to the cold wall through the liquid. This means that conduction through vapor is negligible compared with heat release in condensation. This situation is typical for the films with thickness substantially smaller than the equilibrium film thickness. We expect, therefore, that one-sided model provides a good approximation for the initial stage of growth of relatively thin films.

The right side of (36) is the rational function, and it is possible to find the analytic solution of (36). Instead of that, we plot the numerical solution and compare this with the one-sided expressions[1] to assess the effects of heat conduction through vapor phase. Results for $\hat{k}=0.035$ (corresponding to vapor–water at normal conditions) are shown on Fig. 3. (Note that the length, time and velocity scales here and in Ref. 1 are different; we omit simple but cumbersome detailed comparisons between the scales.) As may be seen, both models give similar results up to times that $h<1$. Near $h=1$, the influence of the hot wall becomes essential; the heat conducted through vapor prevents the further condensation of the vapor, and the film thickness approaches the steady thickness $h=1$, whereas in one-sided model the film grows asymptotically as in $t^{1/2}$.

Figure 4 shows the ratio of mass fluxes in the generalized model (31) to those in the one-sided case (37) versus h for various $\hat{k}$. For small $\hat{k}$ and relatively small h, the ratio is close to one, and therefore, the one-sided model gives a very good simple approximation. In this case, the main mechanism of the heat transfer is the conduction of the heat released during the condensation to the cold wall. For h to 1, the interface temperature approaches T_S, and mass transfer

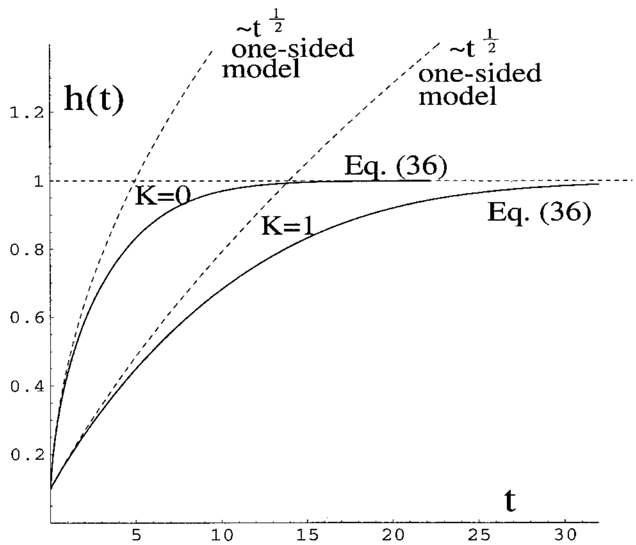

FIG. 3. Comparison of the one-sided model with solution of (36) for $K=1$, $E=0.1$, $h(0)=0.1$, $\hat{k}=0.035$, $T_H=1$.

between the phases decreases. In this case, the prevailing mechanism of heat transport is the heat conduction through the vapor and liquid.

Figure 5 shows the ratio of mass fluxes for various K. It turns out that ratio depends very weakly K and hence the range of validity of one-sided model does not depend significantly on K.

Note that if the film is initially thicker than the equilibrium film, its thickness will diminish and approach the equilibrium thickness, an effect that the one-sided model does not capture.

V. NONLINEAR STABILITY OF THE FILM

The interfacial dynamics will enhance the heat transport between the walls. To describe this increase, we shall derive an evolution equation for the free surface of the condensed film.

For sufficiently large h, the nonlinear term $h^2 h_z$ in Eq. (35) will lead to the breaking of the waves, and to avoid this (35) must be extended to include the vapor recoil and capillary effects.

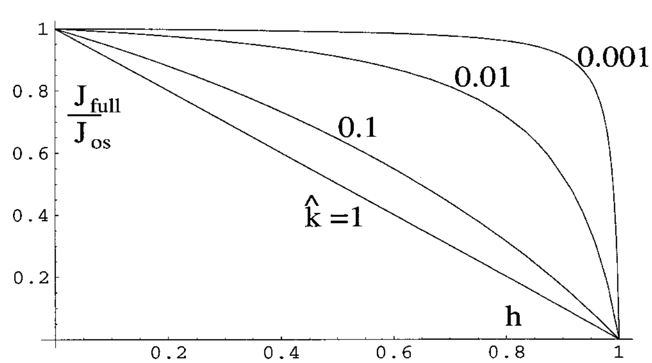

FIG. 4. Ratio of the heat fluxes in full and one-sided model versus h for various $\hat{k}$ for $E=0.1$, $K=1$, $T_H=10$. Here J_{full} is the heat flux in two-sided model, and J_{os} is the heat flux in one-sided model.

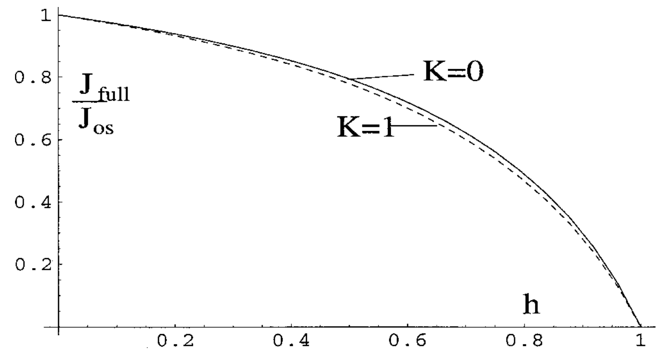

FIG. 5. Ratio of the heat fluxes in full and one-sided model versus h for various K for $E=0.1$, $\hat{k}=0.035$, $T_H=10$. Here J_{full} is the heat flux in two-sided model, and J_{os} is the heat flux in one-sided model.

In the Benney and Kuramato–Sivashinsky models for the isothermal homogenous film $R=O(1)$ and $S \gg 1$.[7,8] Both conditions ensure that the conventional isothermal film flows are weakly unstable, and therefore the waves appearing at film surface are long. As a result, the conventional tools for the derivation of the associated nonlinear evolution equations are (i) the long-wavelength expansions by Benney,[9] or (ii) asymptotic expansions by Nepomnyashchy.[7] The equations derived by those techniques allow one to investigate the finite-amplitude effects of the underlying interfacial dynamics.

In the absence of the temperature variations, the vertically falling film is always unstable, i.e., the critical Reynolds number R_c for the case is zero. This is not the case when condensation is present; here both vapor recoil and the Marangoni effect are stabilizing and hence the critical Reynolds number for the film flowing down a cold wall will be positive, and will depend of the geometrical and physical characteristics of the system. Our subsequent long-wavelength analysis will be valid for the Reynolds number slightly exceeding the critical Reynolds number R_c.

We assume that the dependent variables vary slowly both in time and downstream, so that the lubrication theory may be used. We consider long-wavelength disturbances, and introduce formally the small parameter ε. Introduce new variables as

$$Z=\varepsilon z, \quad X=x, \quad \tau=\varepsilon \tau, \tag{38}$$

and rewrite the governing system of equations and boundary conditions. Further, we expand the dependent variables in power series of ε

$$(w,T,T^{(V)},p,J)=(W,T^{(0)},T^{(V)(0)},p,J) + \varepsilon(w^{(1)},T^{(1)},T^{(V,1)},p^{(1)},J^{(1)})+ \ldots . \tag{39}$$

Here the film thickness $h(z,t)$ is not specified, and considered to be of order of unity. We substitute (39) into the governing equations and boundary conditions, and consider the sequence of problems controlled by the power of parameter ε. In the zeroth order, we will have relations (31)–(34).

Solutions of the problem (31)–(34) depends on the film thickness $h(z,t)$. We need to know only the velocity part of the first-order problem.[10]

First-order problem in ε reads

$$w_{XX}^{(1)} = p_x + R(w_r + w w_Z + u w_X), \quad w^{(1)}(0) = 0,$$
$$w_X^{(1)}(h) = -SC(T_Z + T_X h_Z). \tag{40}$$

We solve (40) for $u^{(1)}$ and substitute $u^{(1)}$ and (31) into (25). This yields the strongly nonlinear equation for the film thickness, which in unscaled variables is

$$h_t - \frac{E}{R} J + h^2 h_z + \frac{S}{3}[h^3 h_{zzz}]_z$$
$$+ \left(\left[\frac{E^2}{3R\hat{\rho}} h^3 \frac{\partial}{\partial h} J^2 + \frac{2}{15} R h^6 + \frac{KCS}{2} h^2 \frac{\partial J}{\partial h} \right] h_z \right)_z = 0. \tag{41}$$

The conductivities ratio $\hat{k}$ is contained in J; see Eq. (31). The e.e. by Burelbach *et al.*[1] may be recovered as a special case of (41) in the limit $\hat{k} \to 0$; in this case J from (31) reduces to $(h+K)^{-1}$, the expression appearing in one-sided model of evaporation. The strongly nonlinear equation (41) is, therefore, the generalization of (10.4) by Burelbach *et al.*[1]

When the film thickness is small the difference between one-sided and two-sided models is negligible. It is interesting, however, to consider the case when the two-sided model differs substantially from one-sided counterpart, namely when the film thickness is close to equilibrium value unity.

Near the equilibrium thickness equation (41) allows a simpler representation. We introduce the interfacial disturbances η, $h = 1 + \eta$, and retain the leading terms of expansion in series of η. The result is

$$\eta_t + 2\eta\eta_z + A\eta + B\eta_{zz} F \eta_{zzzz} = 0, \tag{42}$$

$$A = \frac{E}{R\hat{k}} \Lambda, \quad B = \frac{2}{15} R - \frac{8E^2}{3\hat{\rho}\hat{k}^2 R} \Lambda^2 - \frac{SCK}{2\hat{k}}, \quad F = \frac{S}{3}, \tag{43}$$

$$\Lambda = \frac{1 + \hat{k} T_H}{K + T_H + K T_H} > 0. \tag{44}$$

Equation (42) is a damped KS equation,[11] and accounts for the vapor recoil, Marangoni and wave propagation steepening, and allows one to understand the contribution of these mechanisms in the interfacial film dynamics. The second term in (42) is the kinematic nonlinearity; the third term is due to stabilizing effect of the vapor recoil; the fourth term corresponds to the competition between the destabilizing influence of inertia, and the stabilizing influence of vapor recoil and Marangoni effect, (43); the last term in (43) shows the stabilizing influence of surface tension.

We shall use Eq. (42) to estimate the heat flux through the liquid–vapor system and to evaluate the contribution to the interfacial dynamics of the flux. In the limit $\hat{k} \to 0$, the temperature gradient in the vapor become infinite. Then the very small change of the interface elevation will result in large variations of the interface temperature, which in turn leads to the fast mass transfer. As a result, the formal limit $\hat{k} \to 0$ is not applicable to (42), since it leads to an infinite stabilizing influence of the recoil. Formally, both A and B in (43) become infinite.

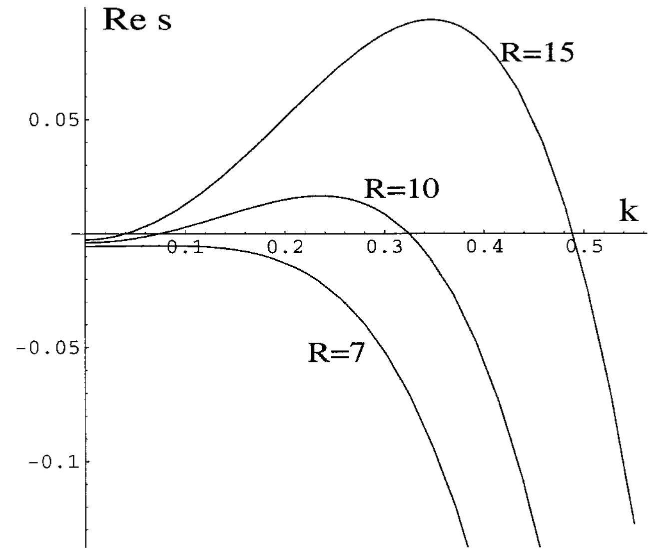

FIG. 6. The growth rate $Re\, s$ versus the wave number k for various Reynolds number R. $\hat{k} = 0.035$; $\hat{\rho} = 6 \cdot 10^{-4}$; $E = 0.01$; $K = 0$; $T_H = 10$; $S = 20$; $R = 7,10,15$.

For small initial film thicknesses the leading mechanism of heat transfer is condensation with heat consumption. For this case Burelbach *et al.*[1] derived the appropriate strongly nonlinear equation for the film thickness.

It is interesting that the impact of Marangoni effect is proportional to CK; C measures the dependence of surface tension on temperature, and K measures the deviation of the surface temperature from the saturation temperature. Note that $C \ll 1$, and in general the third term in (43) is much smaller than the first two terms, corresponding to the main competing mechanisms: Destabilizing by inertia and stabilizing by vapor recoil. However, the difference between these two effects is assumed to be small, and therefore, the small stabilizing impact of Marangoni effect can be comparable with the difference.

In order to understand the typical behavior of solution for (42), consider the linear stability analysis of zeroth basic solution $\eta = 0$ (i.e., unidirectional basic film flow). Note that the *isothermal* falling film is convectively unstable,[8] and relation between temporal and spatial instabilities may be described very well by the Gaster transformation.[12] Since these two approaches are equivalent for small growth rate, we consider the temporal formulation of stability problem. The temporal stability analysis yields

$$\eta \sim e^{ikz+st},$$
$$s = -\frac{E}{R\hat{k}} \Lambda + \left[\frac{2}{15} R - \frac{8E^2}{2\hat{\rho}\hat{k}^2 R} - \frac{SCK}{2\hat{k}} \Lambda \right] k^2 - \frac{S}{3} k^4. \tag{45}$$

The typical plot of instability rate *Re(s)* vs real wave number k is shown on Fig. 6. The simple analysis for critical value B_c of parameter B (and, therefore, for critical Reynolds number) and maximally growing wave number k_c shows that

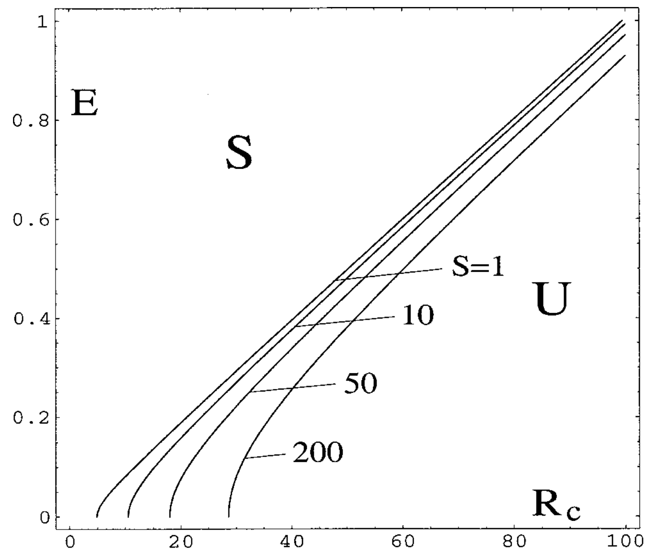

FIG. 7. The marginal stability curves in the plane of critical Reynolds number R_c and evaporation number E for various surface tensions; S and U mark the ranges of stability and instability, respectively.

$$B_c = 2\left(\frac{ES\Lambda}{3R\hat{k}}\right)^{1/2}, \quad k_c = \left(\frac{3E\Lambda}{SR\hat{k}}\right)^{1/4}. \tag{46}$$

Clearly, as S becomes large, k_c becomes small and the asymptotic theory should hold. This instability occurs at finite but long wave numbers, contrary to that in falling films without mass transfer, where for any small k the disturbances are unstable. The stronger surface tension and evaporation number are, weaker the instability. Vice versa, for small E and C, the derivation should be modified to take into account higher-order terms.

For $B > B_c$, the zeroth basic solution of Eq. (42) is unstable. Substitution of (43) in (46) allows one to find critical Reynolds number R_c as a function of the system parameters. Numerical tests for physically reasonable system parameters show that R_c may become substantial, see Fig. 7. On the other hand, density ratio $\hat{\rho}$ is usually very small (about 0.001 for water–vapor at standard conditions), and the stabilizing impact of vapor recoil [proportional to $\hat{k}^{-1}$, second term for expression of B in (43)] becomes an important factor of the stability competition. As a result, for instability the Reynolds number of the film should not be very small in order to overcome the damping effect of vapor recoil, (43) and (46). However, the presented analysis assumes that $R \sim O(1)$; as a result, the above observation might lose its quantitative character for large enough Reynolds numbers.

To understand what happens near the onset of instability, consider the solution of (42) in the form

$$B - B_c = O(\varepsilon^2), \quad \eta = a(z,t)e^{ik_c z}, \quad a(z,t) \sim O(\varepsilon). \tag{47}$$

The result is the Ginzburg–Landau equation for $a(z,t)$:

$$\frac{\partial a}{\partial t} = k_c(B - B_c)a - \frac{32}{9}\frac{1}{B_c}|a|^2 a + 2B_c\frac{\partial^2 a}{\partial x^2}. \tag{48}$$

Since the Landau constant, the coefficient of $|a|^2 a$ is negative, the bifurcation governing the stability of the system is supercritical, and the amplitude of the periodic waves near the bifurcation point equilibrates. This is analogous to the results by Panzarella *et al.*,[3] who found stable finite-amplitude states in the horizontal film boiling in the one-sided model. To understand how the interfacial dynamics affects the averaged heat transport between the walls, assume that a is unmodulated in space, $a = a(t)$, write the expansion of the heat flux H for the small deviations of the interface from its steady-state location

$$\begin{aligned}\langle H\rangle &= H_0 + H_1\langle\eta\rangle + H_2\langle\eta^2\rangle + \ldots\\ &= /\langle\eta\rangle = 0/ = H_0 + H_2\langle\eta^2\rangle + \ldots\\ &\simeq H_0 + \frac{9}{32}H_2 A(B - 2(AF)^{1/2}),\end{aligned} \tag{49}$$

$$H_k = \frac{\partial^{k+1}T^{(L)}}{\partial x^{k+1}} = \frac{1}{k!}\frac{\partial^k \Gamma}{\partial h^k}, \quad \Gamma = \frac{h - 1 - \hat{k}\Omega}{\Sigma}. \tag{50}$$

As a result

$$\langle H\rangle = \frac{h - 1 - \hat{k}\Omega}{\Sigma} + \frac{9}{32}\left[\frac{h - 1 - \hat{k}\Omega}{\Sigma}\right]''_h \times\left[\frac{E\Lambda}{R\hat{k}}\left(\frac{2}{15}R - \frac{8E^2}{3\hat{\rho}\hat{k}^2 R}\Lambda^2 - \frac{SCK}{2\hat{k}}\Lambda - \left(\frac{ES\Lambda}{3R\hat{k}}\right)^{1/2}\right)\right]. \tag{51}$$

We assumed above that the average film elevation is constant, and, therefore, the average of disturbances is zero. We consider quasi-steady dynamics of the falling condensate; therefore, the temperature profile remains piece-wise linear, (31). As a result, the heat flux expressions for H_i may be found from differentiation of Γ, (50), with respect to h.

Relation (51) allows one to find the effective heat flux in the system. Note that in one-sided model the average thickness of the film grows as $t^{1/2}$. Therefore, the basic heat flux, being inversely proportional to the film thickness, decays with time as $t^{-(1/2)}$. In contract to that, expression (51) shows the saturation of the heat flux caused by the influence of the outer wall.

Simple analysis shows that the heat transport due to wavy film motion is enhanced when Reynolds number becomes large, i.e., due to destabilizing inertia influence. From the other hand, the larger evaporation number E results in stronger recoil stabilization impact. The typical plot of H versus E for the film dynamics near the equilibrium thickness is shown on Fig. 8. For the stable condensing film, the surface disturbances are decaying, and the heat flux is just H_0. For the unstable film, the finite-amplitude interfacial waves enhance the heat transport between the walls. Surface tension diminishes the corrugation on the surface, and also reduces the wavy heat transport.

Note that the above analysis is correct only for small-amplitude waves. For larger waves, the temperature field will

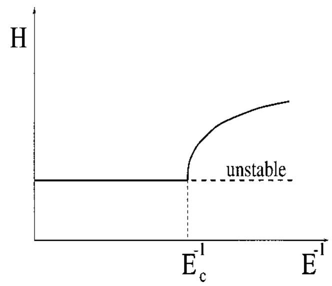

FIG. 8. The typical dependence of heat flux H versus evaporation number E. Here E_c is the critical evaporation number.

be different from the steady one, and waves may even touch the cold walls, creating dry spots on hot wall. These situations require much more elaborated analysis.

VI. CONCLUSION

We have considered the dynamics of condensed film between the hot and cold walls. We retained the equation for the heat transfer through the vapor, and checked the validity of one-sided model of evaporation–condensation proposed by Burelbach *et al.*[1] It turns out that the one-sided model gives very good results when the layer is thinner than the equilibrium thickness. The heat conduction through the vapor becomes important when the layer is thicker. Since there is condensation, and the layers thickens, the one-sided model will break down at long enough times. This contracts with an evaporation layer which thins with time and hence the one-sided model improves with time. We derived the damped KS equation for the interfacial dynamics of the film, and found how the heat transfer depends on the system parameters.

ACKNOWLEDGMENT

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