Finite Element Method for Purely Viscous Flow Problems

In this chapter, we discuss the application of the finite element method to solve purely viscous flow problems. We assume the flow is incompressible and isothermal. Under these conditions, the momentum and mass balance equations take the following form in tensor notation.

*Momentum Equation:*
\[
\rho \left[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\nabla p + \nabla \cdot \mathbf{\tau} + \rho \mathbf{g}
\]

*Continuity Equation:*
\[
\nabla \cdot \mathbf{v} = 0
\]

The rheology of the fluid is contained in the extra-stress tensor \( \mathbf{\tau} \), which describes the stress-strain relationship of the fluid. Recall that a purely viscous flow is one in which the main rheological character of the fluid is given by the viscosity law:

**Purely Viscous Fluid:** \( \eta = \eta(\dot{\gamma}) \) where
\[
\dot{\gamma} = \left| \dot{\mathbf{\gamma}} \right| = \sqrt{\frac{1}{2} \dot{\mathbf{\gamma}} : \dot{\mathbf{\gamma}}^T} = \sqrt{\frac{1}{2} \dot{\gamma}_{ij} \dot{\gamma}_{ij}}
\]

is the shear rate, or magnitude of the rate-of-strain tensor
\[
\dot{\gamma} = \nabla \mathbf{v} + (\nabla \mathbf{v})^T.
\]

The constitutive equation for a purely viscous fluid is
\[
\mathbf{\tau} = \tau(\dot{\gamma}) = \eta(\dot{\gamma}) \dot{\gamma}.
\]

There is no elastic character to the fluid (i.e. we exclude viscoelastic fluids). Purely viscous fluids include both Newtonian fluids and the class of (inelastic) non-Newtonian fluids.

For a Newtonian fluid, the viscosity \( \eta \) is constant at a given temperature, and the constitutive equation is:

**Newtonian Constitutive Equation:** \( \mathbf{\tau} = \mu \dot{\gamma} \) where \( \mu \) is the constant viscosity.

For an (inelastic) non-Newtonian fluid, the viscosity depends on the shear rate and the constitutive equation is:

**(Inelastic) Non-Newtonian Constitutive Equation:** \( \mathbf{\tau} = \eta(\dot{\gamma}) \dot{\gamma} \)

We can think of the inelastic non-Newtonian constitutive equation as being a generalization of the constitutive equation for a Newtonian fluid in which the constant viscosity \( \mu \) is replaced by the shear-rate dependent viscosity \( \eta = \eta(\dot{\gamma}) \). For this reason, inelastic non-Newtonian fluids (or purely viscous fluids) are also called *Generalized Newtonian fluids*.

Conversely, we can think of the Newtonian constitutive equation as a special case of the (inelastic) non-Newtonian constitutive equation, in which we take the viscosity to be constant, i.e. \( \eta(\dot{\gamma}) = \mu \). In fact, both types of fluids can be treated similarly when applying the finite element method. We will see that the non-Newtonian case simply introduces an additional nonlinearity, but that no new numerical concepts arise. Therefore, in our application of the FEM, we will use the Generalized Newtonian constitutive equation and indicate how equations are changed for the Newtonian case, where \( \eta(\dot{\gamma}) = \mu \).
Below we give the full flow problem (in tensor notation) we want to solve for a purely viscous fluid, assuming incompressible, isothermal conditions:

**Viscous Flow Problem:**

\[
\text{Momentum Equation:} \quad \mathbf{F} = -\nabla p + \mathbf{F}_{\tau} - \rho \left( \frac{d}{d t} \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) + \rho g = 0 \quad \text{on} \quad \Omega \quad (5)
\]

\[
\text{Continuity Equation:} \quad \nabla \cdot \mathbf{v} = 0 \quad \text{on} \quad \Omega \quad (6)
\]

\[
\text{Constitutive Equation:} \quad \mathbf{F}_{\tau} = \eta \mathbf{D} \quad \text{on} \quad \Omega \quad (7)
\]

\[
\text{Boundary Conditions:} \quad \mathbf{v} = \mathbf{v}_0 \quad \text{on} \quad \Gamma_v \quad \text{(essential boundary condition)} \quad (8)
\]

\[
\mathbf{t} = \mathbf{t}_0 \quad \text{on} \quad \Gamma_t \quad \text{(natural boundary condition)} \quad (9)
\]

where: \( \Omega \) is the domain in \( \mathbb{R}^2 \) (for 2-dimensional problems) or \( \mathbb{R}^3 \) (for 3-dimensional problems) on which we solve the field equations,

\( \Gamma_v \) and \( \Gamma_t \) represent portions of the domain boundary \( \partial \Omega \) such that \( \Gamma_v \cap \Gamma_t = \emptyset \) (i.e. \( \Gamma_v \) and \( \Gamma_t \) do not overlap, or intersect) and \( \partial \Omega = \Gamma_v \cup \Gamma_t \).

\( \mathbf{F} = -\rho \mathbf{a} + \mathbf{F}_{\tau} \) is the *total stress tensor* (or *Cauchy stress tensor*),

\( \mathbf{t} = \mathbf{F} \cdot \mathbf{n} \) is the *traction vector*,

\( \mathbf{v} \) represents specified values of velocity on the portion of the domain boundary represented by \( \Gamma_v \),

\( \mathbf{t} \) represents specified traction values on the portion of the domain represented by \( \Gamma_t \).

We make the following two comments concerning the above **Viscous Flow Problem**.

**Comments:**

1. The terms \( \nabla \cdot \mathbf{F}_{\tau} = \nabla \cdot \{ \eta (\dot{\gamma}) \dot{\gamma} \} = \nabla \cdot \{ \eta (\dot{\gamma}) \left[ \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] \} \) contain second-order derivatives of velocity components. Therefore, by substitution of the constitutive equation into the momentum equation, there are second-order derivatives of velocity components in the momentum equation.

2. If \( \Gamma_t = \emptyset \), so that \( \Gamma_v = \partial \Omega \) and velocity is imposed everywhere on the boundary \( \partial \Omega \), then pressure appears only as a derivative in the problem, so that pressure would only be determined up to an additive constant. To avoid this indeterminacy, we need an additional constraint on pressure. We can either (i) impose a value of pressure at one nodal point (this would constitute an additional essential boundary condition), or (ii) in our weak formulation of the problem, choose the space \( L^2_0(\Omega) \) for pressure, instead of \( L^2(\Omega) \), where \( L^2_0(\Omega) = \left\{ \mathbf{v} \in L^2(\Omega) : \int_{\Omega} \mathbf{v} \ d\Omega = 0 \right\} \) (i.e. force pressure to have zero mean over the domain \( \Omega \)).
Compared to our previous experience with applying the FEM, there are several new aspects or difficulties associated with the **Viscous Flow Problem**. These are given below:

**New Aspects / Difficulties:**

1. There is more than one equation to be solved on the domain and more than one unknown function. 
   - **In 2-dimensions:** 2 momentum equations and 1 continuity equation; 
     2 unknown velocity components, \( v_1 \) and \( v_2 \), and an unknown pressure function, \( p \).
   - **In 3-dimensions:** 3 momentum equations and 1 continuity equation; 
     3 unknown velocity components, \( v_1 \), \( v_2 \) and \( v_3 \), and an unknown pressure function, \( p \).
   
   *Note:* The extra-stress \( \tau \) is not a primary unknown. Its values are determined directly from a given velocity field.

The two most common approaches for solving the **Viscous Flow Problem** are:

- **Mixed formulation** in which velocity and pressure are treated as the primary unknowns;
- **Penalty method formulation**, in which pressure is eliminated as an unknown by replacing the incompressibility condition (i.e. continuity equation) by a so-called penalty equation for pressure of the form \( p = -\frac{1}{\varepsilon}(\nabla \cdot \mathbf{u}) \), where \( \varepsilon \ll 1 \) is some small penalty parameter. This penalty equation has the effect of slightly relaxing the incompressibility condition.

2. The continuity equation serves as a scalar constraint on the velocity. It is referred to as the incompressibility constraint. It is not a trivial matter to handle or enforce the incompressibility constraint in FEM. The FEM approximation we choose for velocity must satisfy the incompressibility constraint in some sense.

3. If a mixed FEM formulation is used, then an “appropriate” choice of finite elements is needed for velocity and pressure. That is, we cannot choose the velocity and pressure approximations independently, combining any finite element for pressure with any finite element for velocity. The choice of the two FEM spaces are related.

4. There are two possible sources of nonlinearities:
   - Convection (or inertia) terms \( \rho \mathbf{u} \cdot \nabla \mathbf{u} \).
   - Non-Newtonian constitutive equation \( \tau = \eta(\dot{\gamma})\dot{\gamma} \).

*Notes:*

- Stokes flow (for a Newtonian fluid) is a linear problem; the momentum equation is \( -\nabla p + \mu \nabla^2 \mathbf{u} = 0 \). It contains neither of the nonlinearities given above; it contains no inertia terms (since \( Re = 0 \)) and the Newtonian viscosity law is linear (since \( \eta(\dot{\gamma}) = \mu = const. \)).
- The Navier-Stokes equations are nonlinear. They contain the first nonlinearity since the inertia terms \( \rho \mathbf{u} \cdot \nabla \mathbf{u} \) are present, but not the second nonlinearity.
- The equations for the creeping flow of an (inelastic) non-Newtonian fluid are nonlinear. The equations contain the second source of nonlinearity, but not the first type of nonlinearity (since creeping flow means that \( Re = 0 \), so that the inertia terms may be neglected).
- The equations for the noncreeping flow of an (inelastic) non-Newtonian fluid are nonlinear. They contain both types of nonlinearities mentioned above.
FEM for 2-Dimensional Planar Flow

We first consider the 2-dimensional planar flow of a purely viscous fluid. As previously stated, we assume incompressible flow and isothermal conditions. Presently, we also assume the flow to be steady, so that no time derivatives exist. Modifications to the formulation derived here will be made at later points to solve:

1. Transient problems, by including the time-derivative terms for velocity in the momentum equation;
2. Three-dimensional problems;
3. Axisymmetric flow problems (or flow problems in cylindrical coordinates);
4. Nonisothermal flow problems, by including the energy equation in the system of equations.

Two-dimensional planar flow implies the formulation of the Viscous Flow Problem in the Cartesian, or rectangular, coordinate system, using the rectangular coordinates \((x, y)\). The velocity vector \(\mathbf{v} = \mathbf{v}(x, y)\) has two components and is denoted by

\[
\mathbf{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}
\]

(10)

where \(u = u(x, y)\) and \(v = v(x, y)\). The rate-of-strain tensor and shear rate take the following form in 2-dimensional, planar flow:

\[
\dot{\gamma} = \nabla \mathbf{v} + (\nabla \mathbf{v})^T = \begin{bmatrix} 2u_{,x} & u_{,y} + v_{,x} \\ u_{,y} + v_{,x} & 2v_{,x} \end{bmatrix}
\]

(11)

\[
I_2 \equiv \dot{\gamma} = \sqrt{\frac{1}{2} \dot{\gamma}_{ij} \dot{\gamma}_{ij}} = \left[ 2u_{,x}^2 + 2v_{,y}^2 + (u_{,y} + v_{,x})^2 \right]^{1/2}.
\]

(12)

We will henceforth use the notation \(I_2\) to represent the shear rate \(\dot{\gamma}\). This illustrates the fact that \(\dot{\gamma}\) is an invariant of the rate-of-strain tensor \(\dot{\gamma}\), that is, the value of \(I_2 \equiv \dot{\gamma}\) is independent of the coordinate system in which the components of \(\dot{\gamma}\) are represented.
The system of equations we want to solve may be written as:

**System (I):**

**Momentum Equations:**

\[ \tau_{xx,x} + \tau_{yx,y} - p_{,x} - \rho(uu_{,x} + vu_{,y}) + \rho g_x = 0 \]  
\[ \tau_{xy,x} + \tau_{yy,y} - p_{,y} - \rho(\nu v_{,x} + \nu v_{,y}) + \rho g_y = 0 \]  

**Continuity Equation:**

\[ u_{,x} + v_{,y} = 0 \]

**Constitutive Equations:**

\[ \tau_{xx} = 2\eta(I_2) u_{,x} \]  
\[ \tau_{yy} = 2\eta(I_2) v_{,y} \]  
\[ \tau_{xy} = \eta(I_2) [u_{,y} + v_{,x}] = \tau_{yx} \]

**Boundary Conditions:**

\[ \left\{ \begin{array}{l} u = \tilde{n} \\ v = \tilde{v} \end{array} \right\} \text{ on } \Gamma_v \quad (\text{Essential b.c.}) \]  
\[ \left\{ \begin{array}{l} t_x \equiv (-p + \tau_{xx})n_x + \tau_{yx} n_y = \tilde{t}_x \\ t_y \equiv \tau_{yx} n_x + (-p + \tau_{yy}) n_y = \tilde{t}_y \end{array} \right\} \text{ on } \Gamma_t, \quad (\text{Natural b.c}) \]

We rewrite System (I) in the following equivalent form:

**System (I E):**

Find \( u, v \in H^2(\Omega) \) and \( p \in H^1(\Omega) \) such that for all \( w, q \in L^2(\Omega) \),

\[ (\tau_{xx,x} + \tau_{yx,y} - p_{,x} ; w) - (\rho[uu_{,x} + vu_{,y}] ; w) + (\rho g_x ; w) = 0 \]  
\[ (\tau_{xy,x} + \tau_{yy,y} - p_{,y} ; w) - (\rho[\nu v_{,x} + \nu v_{,y}] ; w) + (\rho g_y ; w) = 0 \]  
\[ (u_{,x} + v_{,y} ; q) = 0 \]

subject to the constitutive equations, Eq. (16)-(18), and boundary conditions, Eqs (19) and (20).

We have used the notation: \((f ; g) = \int f g d\Omega\).

**Note:** Second derivatives of \( u \) and \( v \) are contained in the (first) derivatives of \( \tau \).

**Note:** Without loss of generality, we may assume homogeneous velocity boundary conditions in Eq. (19);

that is, \( \tilde{u} = 0 \) and \( \tilde{v} = 0 \).
System \((I^E)\) is the starting point for the **Method of Weighted Residual**. We get the weak formulation by restricting the space to which the weight functions \(w\) belong, thus enlargening the spaces to which \(u, v\) and \(p\) belong. Specifically, we take

\[
 w \in H^1_I(\Omega) \subseteq L^2(\Omega), \text{ where } H^k_I(\Omega) = \left\{ u \in H^k(\Omega) : u|_\Gamma = 0 \right\}.
\]

By taking \(w \in H^1_I(\Omega)\) (specifically, by taking \(w \in H^1(\Omega)\)), we may perform integration by parts on the integrals involving the stress and pressure in System \((I^E)\) via the Divergence Theorem.

**Note:** In general,

\[
 \int_{:\Omega} (-\nabla p + \nabla \cdot \tau) w \, d\Omega = \left[ \nabla \cdot \left( -p \tilde{\Omega} + \tau \right) \right] w \, d\Omega
\]

\[
 = \int_{\partial\Omega} w \left( -p \tilde{\Omega} + \tau \right) \cdot n \, d\Gamma - \int_{\Omega} \left( -p \tilde{\Omega} + \tau \right) \cdot \nabla w \, d\Omega
\]

(by Divergence Theorem)

\[
 = \int_{\Gamma_t} \int_{\Omega} (-p \tilde{\Omega} + \tau) \cdot n \, d\Gamma - \int_{\Omega} \left( -p \tilde{\Omega} + \tau \right) \cdot \nabla w \, d\Omega
\]

(since \(w = 0\) on \(\Gamma_v\))

\[
 = \int_{\Gamma_t} \int_{\Omega} \tau \cdot n \, d\Gamma - \int_{\Omega} \left( -p \tilde{\Omega} + \tau \right) \cdot \nabla w \, d\Omega
\]

(since \(\tau \equiv (-p \tilde{\Omega} + \tau) \cdot n\) on \(\Gamma_t\))

In our 2-dimensional planar flow problem, the above may be written as:

\[
 (-p_{xx} + \tau_{xx,x} + \tau_{xy,y} \cdot w) = -(-p + \tau_{xx} \cdot w_{x} + \tau_{xy} \cdot w_{y}) + \int_{\Gamma_t} \tau_{w} \, d\Gamma \tag{24}
\]

\[
 (-p_{yx} + \tau_{yx,x} + \tau_{yy,y} \cdot w) = -(-p + \tau_{yx} \cdot w_{x} + \tau_{yy} \cdot w_{y}) + \int_{\Gamma_t} \tau_{y} \, d\Gamma \tag{25}
\]

Substituting Eqs (24) and (25) into Eqs (21) and (22) in System \((I^E)\) yields the following **(Continuous) Weak Formulation**:
System (II):

Find \( u, v \in H^1_{\Gamma_v}(\Omega) \) and \( p \in L^2(\Omega) \) such that for all \( w \in H^1_{\Gamma_v}(\Omega) \) and \( q \in L^2(\Omega) \), we have

\[
(-p + \tau_{xx} ; w_{,x}) + (\tau_{yy} ; w_{,y}) + \rho(uu_{,x} + vv_{,y} ; w) = \rho\left(g_x ; w\right) + \int_{\Gamma_v} \bar{t}_x w d\Gamma \\
(\tau_{xy} ; w_{,x}) + (-p + \tau_{yy} ; w_{,y}) + \rho(uv_{,x} + vv_{,y} ; w) = \rho\left(g_y ; w\right) + \int_{\Gamma_v} \bar{t}_y w d\Gamma \\
(u_{,x} + v_{,y} ; q) = 0
\]

subject to the constitutive equations, Eqs (16)-(18).

Notes:

1. Due to the integration by parts on the integrals containing \( \tau \) and \( p \), System (II) contains only first partial derivatives of \( u \) and \( v \) and no derivatives of \( p \). Therefore, we need only require that

\[
u, v \in H^1(\Omega) \quad \text{and} \quad p \in L^2(\Omega)
\]

Thus, we have enlarged the spaces to which velocity and pressure belong. In other words, the integration by parts, which was allowed by our restriction of the weight functions \( w \) to \( H^1_{\Gamma_v}(\Omega) \), allowed us to relax the (rather strong) continuity requirements on \( u \) and \( v \) (and \( p \)) and to construct \( u, v \) and \( p \) with a much wider class of functions when looking for an approximation.

2. We further take \( u, v \in H^1_{\Gamma_v}(\Omega) \) to reflect our assumption of homogeneous Dirichlet boundary conditions in Eq. (5) (see previous note). We may do this without loss of generality; we assume we are solving the auxiliary problem previously described when discussing how to handle different types of boundary conditions.

3. If \( \Gamma_t = \emptyset \), so that \( \Gamma_v = \partial \Omega \) and velocity is imposed everywhere on the boundary \( \partial \Omega \), then pressure appears only as a derivative in the original problem (System (I)), so that pressure would only be determined up to an additive constant. To avoid this indeterminacy, we need an additional constraint on pressure. We can either (i) impose a value of pressure at one nodal point (this would constitute an additional essential boundary condition), or (ii) choose the space \( L^2_0(\Omega) \) for pressure, instead of \( L^2(\Omega) \), where \( L^2_0(\Omega) = \left\{ q \in L^2(\Omega) : \int_{\Omega} q d\Omega = 0 \right\} \) (i.e. force pressure to have zero mean over the domain \( \Omega \)).

4. We have used the weak form of the natural boundary conditions in System (II):

\[
\int_{\Gamma_t} t_x w d\Gamma = \int_{\Gamma_t} \bar{t}_x w d\Gamma \quad \text{and} \quad \int_{\Gamma_t} t_y w d\Gamma = \int_{\Gamma_t} \bar{t}_y w d\Gamma.
\]
We now apply the Galerkin Method to System (II):

Let $S_v^h \subset H^1_{\Gamma_v}(\Omega)$ be the finite-dimensional subspace of the (infinite-dimensional) space $H^1_{\Gamma_v}(\Omega)$ on which we approximate the functions $u$ and $v$, and let $u^h, v^h \in S_v^h$ be the approximations to $u$ and $v$ from $S_v^h$. Similarly, let $S_p^h \subset L^2(\Omega)$ be the finite-dimensional subspace of the (infinite-dimensional) space $L^2(\Omega)$ on which we approximate the function $p$, and let $p^h \in S_p^h$ be the approximation to $p$ from $S_p^h$. Let $\left\{ \psi_i^v \right\}_{i=1}^{N_v}$ be a basis for $S_v^h$ consisting of $N_v$ global shape functions (which form a basis for $S_v^h$), and let $\left\{ \psi_i^p \right\}_{i=1}^{N_p}$ be a basis for $S_p^h$ consisting of $N_p$ global shape functions (which form a basis for $S_p^h$). Since these two sets of global shape functions form a basis for $S_v^h$ and $S_p^h$, we write the approximations $u^h, v^h \in S_v^h$ and $p^h \in S_p^h$ as linear combinations of $\psi_i^v$ and $\psi_i^p$, respectively:

$$
\begin{align*}
  u^h(x) &= \sum_{i=1}^{N_v} u_i \psi_i^v(x), & v^h(x) &= \sum_{i=1}^{N_v} v_i \psi_i^v(x), & p^h(x) &= \sum_{i=1}^{N_p} p_i \psi_i^p(x)
\end{align*}
$$

(29)

where the $u_i, v_i$ and $p_i$ are constants.

Now, solving System (II) on the finite-dimensional spaces $S_v^h$ and $S_p^h$ yields the discrete weak formulation of our problem.

Since $S_v^h$ and $S_p^h$ have finite dimension, it is equivalent to say that System (II) is satisfied whenever $w = \psi_i^v$ and $q = \psi_i^p$.

(This is the same as writing $w^h = \sum_{i=1}^{N_v} w_i \psi_i^v$ and $q^h = \sum_{i=1}^{N_p} q_i \psi_i^p$ and then factoring out the $w_i$ and $q_i$ constants.)
Galerkin Form:

System (IIₜ):

Find $u^h, v^h \in S^h_v$ and $p^h \in S^h_p$ such that

\[
(-p^h + 2\eta(I^h_2)u^h_{,x} ; \psi^v_{i,x}) + (\eta(I^h_2)[u^h_{,y} + v^h_{,x}] ; \psi^v_{i,y}) \\
+ \rho(u^h_{,x} v^h + v^h_{,x} u^h_{,y} ; \psi^v_{i}) = \rho(g^v_{,x} ; \psi^v_{i}) + \int_{\Gamma^v} \psi^v_{i} d\Gamma
\]

(30)

\[
(\eta(I^h_2)[u^h_{,y} + v^h_{,x}] ; \psi^v_{i,x}) + (-p^h + 2\eta(I^h_2)v^h_{,y} ; \psi^v_{i,y}) \\
+ \rho(u^h_{,y} v^h + v^h_{,y} u^h_{,x} ; \psi^v_{i}) = \rho(g^v_{,y} ; \psi^v_{i}) + \int_{\Gamma^v} \psi^v_{i} d\Gamma
\]

(31)

\[
(u^h_{,x} + v^h_{,y} ; \psi^p_{k}) = 0
\]

(32)

for $i, j = 1, \ldots, N_v$ and $k = 1, \ldots, N_p$, except:

If a (homogeneous) essential boundary condition (homogeneous Dirichlet boundary condition) is imposed at node $l$ for velocity, then the corresponding Galerkin equations are replaced by the boundary condition.

That is, if node $l$ with coordinates $\vec{x}_l$ is such that $\vec{x}_l \in \Gamma^v$, where we apply the boundary condition $u = \bar{u} \equiv 0$ and $v = \bar{v} \equiv 0$ on $\Gamma^v$, then we replace the Galerkin equations corresponding to node $l$ with $u_l = 0$ and $v_l = 0$.

Note: $I^h_2 = \left[2u^h_{,x} + 2v^h_{,y} + (u^h_{,y} + v^h_{,x})^2\right]^{1/2}$

(33)
Substituting the expressions for \( u^h, v^h \) and \( p^h \) into System (II\(^h\)) produces the following nonlinear system of algebraic equations for 2-dimensional planar flow of a purely viscous fluid, under the assumptions of steady-state, incompressible and isothermal conditions:

\[
\begin{align*}
A^1 u + B v + F u - C^1 p &= b^x \\
B^T u + A^2 v + F v - C^2 p &= b^y \\
(C^1)^T u + (C^2)^T v &= 0
\end{align*}
\]

(34a) \hspace{1cm} (34b) \hspace{1cm} (34c)

where:

\[
\begin{align*}
&u^h, v^h \text{ are } N_v \times 1 \text{ column vectors containing the coefficients } u_i, v_i \text{ in the expansions} \\
&\quad u^h = \sum_{i=1}^{N_v} u_i \psi^y_i \text{ and } v^h = \sum_{i=1}^{N_v} v_i \psi^y_i
\end{align*}
\]

where the \( \psi^y_i \) are the global shape functions for velocity;

\[
\begin{align*}
p^h \text{ is a } N_p \times 1 \text{ column vector containing the coefficients } p_i \text{ in the expansion} \\
&\quad p^h = \sum_{i=1}^{N_p} p_i \psi^p_i
\end{align*}
\]

where the \( \psi^p_i \) are the global shape functions for pressure;

\[
\begin{align*}
b_x^y, b_y^y \text{ are } N_v \times 1 \text{ column vectors containing external forces and boundary conditions with components} \\
&\quad b_x^y = \rho (f_x ; \psi^y_i) + \int_{\Gamma_x} \psi^y_i d\Gamma \\
&\quad b_y^y = \rho (f_y ; \psi^y_i) + \int_{\Gamma_y} \psi^y_i d\Gamma
\end{align*}
\]

(35)

\[
\begin{align*}
A^{1}_{ij} &= (2 \eta(I_2^h) \psi^y_{j,x} ; \psi^y_{i,x}) + (\eta(I_2^h) \psi^v_{j,y} ; \psi^v_{i,y}) \\
A^{2}_{ij} &= (\eta(I_2^h) \psi^v_{j,x} ; \psi^v_{i,x}) + (2 \eta(I_2^h) \psi^v_{j,y} ; \psi^v_{i,y}) \\
B_{ij} &= (\eta(I_2^h) \psi^v_{j,x} ; \psi^v_{i,y})
\end{align*}
\]

(36a) \hspace{1cm} (36b) \hspace{1cm} (36c)
\[ F \] is a \( N_v \times N_v \) matrix containing the convection terms with components

\[
F_{ij} = \rho(D_{ijkl} u_j + E_{ijkl} v_j)
\]

where

\[
D_{ijkl} = (\psi_j^v \psi_{k,x} ; \psi_i^v)
\]

and

\[
E_{ijkl} = (\psi_j^v \psi_{k,y} ; \psi_i^v)
\]

(37)

(38)

\( C^1 \) and \( C^2 \) are \( N_v \times N_p \) (divergence) matrices with components

\[
C_{ik}^1 = (\psi_k^p ; \psi_{i,x}^v) \quad \text{and} \quad C_{ik}^2 = (\psi_k^p ; \psi_{i,y}^v)
\]

(39)

\[ \tilde{0} \] is the \( N_v \times 1 \) zero column vector (i.e. \( N_v \times 1 \) column vector containing all zeros).

Comments:

1. \( A^1 \), \( A^2 \), and \( B \) are functions of velocity \( u \) and \( v \) whenever the viscosity \( \eta \) is not constant, i.e. when the flow is non-Newtonian. In this case, the diffusion terms in the algebraic system are nonlinear.

2. The convection terms \( F \) are functions of velocity \( u \) and \( v \), and hence represent nonlinear terms in the algebraic system. These terms vanish when the Reynolds number is zero, i.e. for Stokes, or creeping, flow.

3. The divergence matrices \( C^1 \) and \( C^2 \) do not depend on velocity; they are linear terms.

4. The discrete incompressibility constraint is a linear system of \( N_p \) equations in \( 2N_v \) unknown.

5. This system of equations was derived assuming homogeneous Dirichlet boundary conditions on velocity over part of the domain, i.e. \( u=v=0 \) on \( \Gamma_v \). When these boundary conditions are inhomogeneous, i.e. \( u=\tilde{u} \) and \( v=\tilde{v} \) on \( \Gamma_v \), then we would first construct two functions \( u_0 \) and \( v_0 \) such that \( u_0=\tilde{u} \) and \( v_0=\tilde{v} \) on \( \Gamma_v \) and take the approximations

\[
u^h = u_0 + \sum_{i=1}^{N_v} u_i \psi_i^v \quad \text{and} \quad v^h = v_0 + \sum_{i=1}^{N_v} v_i \psi_i^v
\]

(40)

instead of the ones previously given (with the \( u_0 \) and \( v_0 \) terms omitted). We would then get additional terms on the right hand side of the system of equations.
The system of algebraic equations may also be written as:

\[
\hat{\lambda}(\hat{u})\hat{u} + \hat{F}(\hat{u})\hat{u} + \hat{C}\hat{p} = \hat{b}
\]
(41)

\[
\hat{C}^T\hat{u} = 0
\]
(42)

where

\[
\hat{u} = \begin{bmatrix} u \\ v \end{bmatrix}
\]
is a \(2N_v\times1\) column vector;

\[
p
\]
is the same \(N_p\times1\) column as before;

\[
\hat{b} = \begin{bmatrix} b^x \\ b^y \end{bmatrix}
\]
is a \(2N_v\times1\) column vector;

\[
\hat{A} = \hat{A}(\hat{u}) = \begin{bmatrix} A^1 & B \\ B^T & A^2 \end{bmatrix}
\]
is the \(2N_v\times2N_v\) diffusion matrix;

\[
\hat{F} = \hat{F}(\hat{u}) = \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix}
\]
is the \(2N_v\times2N_v\) convection matrix, where, here, \(0\) is the \(N_v\timesN_v\) zero matrix;

\[
\hat{C} = \begin{bmatrix} -C^1 \\ -C^2 \end{bmatrix}
\]
is the \(2N_v\timesN_p\) divergence matrix.

Various methods can be used to solve this nonlinear system of equations using iterative techniques. Some widely used methods will be discussed at a later point.
1.3. LBB Stability Condition

Unstable Elements:

**Element I: P$_1$-P$_0$ Triangle**

![Diagram of P$_1$-P$_0$ Triangle]

The finite element spaces for velocity and pressure are given by:

- **Velocity degrees of freedom**
  
  
  \[ S_v^h = \left\{ v : v|_{\Omega^e} \in P_1(x, y), \quad \forall \Omega^e \in T^h ; \quad v \in C^0(\Omega) ; \quad v \text{ satisfies boundary conditions} \right\} \]

- **Pressure degree of freedom**
  
  \[ S_p^h = \left\{ q : q|_{\Omega^e} \in P_0(x, y), \quad \forall \Omega^e \in T^h ; \quad \int_{\Omega} q d\Omega = 0 \right\} \]

Velocity is an approximation of the $P_1 - C^0$ type; that is, on each triangular element in the triangulation, or mesh, each component of velocity $v^h_i$ is approximated by a polynomial of the form $v^h_i(x, y) = a_i x + b_i y + c_i$, where the $a_i$, $b_i$ and $c_i$ are constants which differ from element to element, but are such that $v^h_i$ is continuous across element boundaries, and hence over the whole domain. Pressure is an approximation of the $P_0 - C^{-1}$ type; that is, on each triangular element of the mesh, pressure is approximated by a constant. Necessarily, then, pressure is discontinuous across element boundaries.

This element does not satisfy the LBB stability condition. The reason is that, depending on the boundary conditions, it is possible that the discrete continuity equation, $b(v^h, q^h) = 0$ for all $q^h \in S_p^h$, implies that $v^h = 0$. In other words, the only discretely divergence-free vector belonging to $S_v^h$ is the zero vector. In this sense, the discrete continuity equation over-constrains the system; it imposes too many constraints on the discrete velocity field $v^h \in S_v^h$, i.e. there are not sufficiently many velocity degrees of freedom relative to pressure degrees of freedom.