

A BARBAN-DAVENPORT-HALBERSTAM ASYMPTOTIC FOR NUMBER FIELDS

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ABSTRACT. Let K be a fixed number field, and assume that K is Galois over \mathbb{Q} . Previously, the author showed that when estimating the number of prime ideals with norm congruent to a modulo q via the Chebotarëv Density Theorem, the mean square error in the approximation is small when averaging over all $q \leq Q$ and all appropriate a . In this article, we replace the upper bound by an asymptotic formula. The result is related to the classical Barban-Davenport-Halberstam Theorem in the case $K = \mathbb{Q}$.

1. INTRODUCTION

One of the great results of the 1960s concerning the distribution of primes is that “on average” they are well-distributed in arithmetic progressions. In particular, Barban [1] and, independently, Davenport and Halberstam [2, 3] showed that the square of the error in the Prime Number Theorem for primes in arithmetic progressions is small on average. More precisely, given positive integers a and q , we define the weighted prime counting function $\theta(x; q, a)$ by

$$\theta(x; q, a) := \sum_{\substack{p \leq x \\ p \equiv a \pmod{q}}} \log p.$$

The Prime Number Theorem for primes in arithmetic progressions states that if $\gcd(a, q) = 1$, then

$$\theta(x; q, a) \sim \frac{x}{\varphi(q)}, \tag{1}$$

where $\varphi(q) := \#\{1 \leq a \leq q : \gcd(a, q) = 1\}$ is Euler’s φ -function. The Barban-Davenport-Halberstam Theorem (see [4]) states that, for any fixed $M > 0$,

$$\sum_{q \leq Q} \sum_{\substack{a=1 \\ \gcd(a, q)=1}}^q \left(\theta(x; q, a) - \frac{x}{\varphi(q)} \right)^2 \ll xQ \log x, \tag{2}$$

provided that $x(\log x)^{-M} \leq Q \leq x$. Later, Montgomery [10] and Hooley [7] each gave asymptotic formulations of this result valid for various ranges of Q . Hooley’s method starts with the inequality (2), and so at least implicitly relies on the large sieve. Montgomery’s method, however, is based on a result of Lavrik [9] concerning the distribution of twin primes.

With applications in mind, there have been several generalizations of this result to the integers of a number field. See [6, 13] for example. In [12], the author considered yet another

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generalization of (2) concerning the distribution of prime ideals of a number field. See Theorem 1 below. In the present article, we are concerned with the appropriate asymptotic formulation. See Theorem 2.

2. STATEMENT OF MAIN THEOREM

Let K be a fixed number field. We are concerned with the error in estimating sums of the form

$$\theta_K(x; q, a) := \sum_{\substack{N\mathfrak{p} \leq x, \\ N\mathfrak{p} \equiv a \pmod{q}}} \log N\mathfrak{p}$$

via the Chebotarëv Density Theorem. Here, as usual, \mathfrak{p} denotes a prime ideal of the ring of integers \mathcal{O}_K , and $N\mathfrak{p} := \#(\mathcal{O}_K/\mathfrak{p})$ denotes its norm.

Let ζ_q be a primitive q -th root of unity, and let G_q denote the image of the natural map

$$\mathrm{Gal}(K(\zeta_q)/K) \hookrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q}) \xrightarrow{\sim} (\mathbb{Z}/q\mathbb{Z})^*.$$

In this case, the Frobenius substitution is determined by the value $N\mathfrak{p}$ modulo q ; and the Chebotarëv Density Theorem implies that if $a \in G_q$, then

$$\theta_K(x; q, a) \sim \frac{x}{\varphi_K(q)}, \tag{3}$$

where we have made the definition $\varphi_K(q) := \#G_q = \#\mathrm{Gal}(K(\zeta_q)/K)$.

If we assume further that K/\mathbb{Q} is a Galois extension, then we have the following corollary of Goldstein's generalization of the Siegel-Walfisz Theorem [5]. If $a \in G_q$, then for any fixed $M > 0$,

$$\theta_K(x; q, a) = \frac{x}{\varphi_K(q)} + O\left(\frac{x}{(\log x)^M}\right), \tag{4}$$

provided that $q \leq (\log x)^M$. The following average error result is the main theorem of [12], where we continue to assume that our number field K is a Galois extension of \mathbb{Q} .

Theorem 1. *For a fixed $M > 0$,*

$$\sum_{q \leq Q} \sum_{a \in G_q} \left(\theta_K(x; q, a) - \frac{x}{\varphi_K(q)} \right)^2 \ll xQ \log x$$

if $x(\log x)^{-M} \leq Q \leq x$.

Remark. To be precise, the main theorem of [12] is stated in terms of

$$\psi_K(x; q, a) := \sum_{\substack{N\mathfrak{p}^m \leq x, \\ N\mathfrak{p}^m \equiv a \pmod{q}}} \log N\mathfrak{p}.$$

As usual, the statement and proof of the theorem is virtually unchanged when replacing $\psi_K(x; q, a)$ by $\theta_K(x; q, a)$.

In this article, we continue to assume that K/\mathbb{Q} is Galois and replace the inequality in Theorem 1 by an asymptotic formula. In particular, we show the following.

Theorem 2. For a fixed $M > 0$,

$$\sum_{q \leq x} \sum_{a \in G_q} \left(\theta_K(x; q, a) - \frac{x}{\varphi_K(q)} \right)^2 = [K : \mathbb{Q}] x^2 \log x + C_1 x^2 + O\left(\frac{x^2}{(\log x)^M}\right); \quad (5)$$

and if $1 \leq Q \leq x$,

$$\begin{aligned} \sum_{q \leq Q} \sum_{a \in G_q} \left(\theta_K(x; q, a) - \frac{x}{\varphi_K(q)} \right)^2 &= [K : \mathbb{Q}] x Q \log x - \frac{\varphi(m_K)}{\varphi_K(m_K)} x Q \log(x/Q) + C_2 Q x \\ &+ O\left(x^{3/4} Q^{5/4} + \frac{x^2}{(\log x)^M}\right), \end{aligned} \quad (6)$$

where φ denotes the ordinary Euler φ -function, C_1, C_2 are constants, and m_K is an integer defined in the first paragraph of Section 4.

Remark. The constants C_1, C_2 appearing in the statement of the theorem depend on K and may be given explicitly. However, the expressions are somewhat messy. For example, C_1 is given by

$$C_1 = F(1)\zeta'(2) + F(1)\frac{(2\gamma - 3)\pi^2}{12} + F(1)F'(1)\frac{\pi^2}{6} - [K : \mathbb{Q}].$$

Here, $\zeta(s)$ denotes the Riemann zeta function, $\gamma \approx 0.577$ is the Euler-Mascheroni constant, and $F(s) := h(s) \prod_{\ell | m_K} D_{K, \ell}(s)$. The functions $h(s)$ and $D_{K, \ell}(s)$ are described in Section 4.

Remark. In the case that K/\mathbb{Q} is Abelian, it turns out that $\varphi(m_K)/\varphi_K(m_K) = [K : \mathbb{Q}]$. See the first paragraph of Section 4. Thus, in this case, equation (6) simplifies nicely to

$$\sum_{q \leq Q} \sum_{a \in G_q} \left(\theta_K(x; q, a) - \frac{x}{\varphi_K(q)} \right)^2 = [K : \mathbb{Q}] x Q \log Q + C_2 Q x + O\left(x^{3/4} Q^{5/4} + \frac{x^2}{(\log x)^M}\right).$$

Our proof of Theorem 2 is an adaptation of Hooley's methods for the case $K = \mathbb{Q}$ as found in [7, pp. 209-212]. The proof will be carried out in Section 5.

3. ACKNOWLEDGMENT

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4. PRELIMINARIES

Before proceeding with the proof of Theorem 2, we first analyze the arithmetic function $\varphi_K(q)$. Let $\mathbb{Q}^{\text{cyc}} := \bigcup_{q > 1} \mathbb{Q}(\zeta_q)$, and let $\mathcal{A} := \mathbb{Q}^{\text{cyc}} \cap K$. Then \mathcal{A} is an Abelian extension of \mathbb{Q} of finite degree. In particular, \mathcal{A} is the maximal Abelian subfield of K . By the Kronecker-Weber Theorem, there exists a smallest integer m_K such that $\mathcal{A} \subseteq \mathbb{Q}(\zeta_{m_K})$. See, for example, [8, p. 210]. For each integer $q > 0$, we define the intersection $A_q := K \cap \mathbb{Q}(\zeta_q)$. Whence, via restriction maps, $\text{Gal}(K(\zeta_q)/K) \cong \text{Gal}(\mathbb{Q}(\zeta_q)/A_q)$. Thus, it is clear that if q is coprime to m_K , then $\varphi_K(q) = \varphi(q)$. In any case, $\varphi_K(q)$ is multiplicative and divides $\varphi(q)$. For each prime divisor ℓ of m_K , we define $b_\ell := \text{ord}_\ell(m_K)$, the order of ℓ dividing m_K .

Lemma 1. For a prime ℓ , $\varphi_K(\ell)$ is a divisor of $\ell - 1$. In general, we have

$$\varphi_K(q) = \prod_{\substack{\ell^\alpha \parallel q \\ \ell \nmid m_K}} \ell^{\alpha-1}(\ell - 1) \prod_{\substack{\ell^\alpha \parallel q \\ \ell \mid m_K \\ \alpha \geq b_\ell}} \ell^{\alpha-b_\ell} \varphi_K(\ell) \prod_{\substack{\ell^\alpha \parallel q \\ \ell \mid m_K \\ \alpha < b_\ell}} \varphi_K(\ell).$$

Proof. The first statement is trivial as G_q is a subgroup of $(\mathbb{Z}/q\mathbb{Z})^*$. Since $\varphi_K(q)$ is multiplicative and $\varphi_K(q) = \varphi(q)$ for $\gcd(q, m_K) = 1$, we restrict attention to primes dividing m_K .

Suppose that ℓ is a prime dividing m_K . Then $A_{\ell^{b_\ell+k}} = A_{\ell^{b_\ell}}$ for all integers $k \geq 0$. Thus, we immediately see that

$$\varphi_K(\ell^{b_\ell+k}) = |\text{Gal}(\mathbb{Q}(\zeta_{\ell^{b_\ell+k}})/\mathbb{Q}(\zeta_{\ell^{b_\ell}}))| \cdot |\text{Gal}(\mathbb{Q}(\zeta_{\ell^{b_\ell}})/A_{\ell^{b_\ell}})| = \ell^k \varphi_K(\ell^{b_\ell}). \quad (7)$$

We claim that

$$\varphi_K(\ell^j) = \varphi_K(\ell) \text{ for } 1 \leq j \leq b_\ell. \quad (8)$$

If $b_\ell = 1$, the statement is trivial. Assume then that $b_\ell \geq 2$, and consider the following field diagram.

$$\begin{array}{ccc} & \mathbb{Q}(\zeta_{\ell^{b_\ell}}) & \\ \ell^{b_\ell-1} \swarrow & & \searrow \varphi_K(\ell^{b_\ell}) \\ \mathbb{Q}(\zeta_\ell) & & A_{\ell^{b_\ell}} \\ \searrow \varphi_K(\ell) & & \swarrow \\ & A_\ell & \end{array} \quad (9)$$

Observe that $A_\ell = K \cap \mathbb{Q}(\zeta_\ell) = A_{\ell^{b_\ell}} \cap \mathbb{Q}(\zeta_\ell)$. Since the compositum $A_{\ell^{b_\ell}}\mathbb{Q}(\zeta_\ell)$ is the smallest field containing both $A_{\ell^{b_\ell}}$ and $\mathbb{Q}(\zeta_\ell)$, we have that $\mathbb{Q}(\zeta_{\ell^{b_\ell}}) \supseteq A_{\ell^{b_\ell}}\mathbb{Q}(\zeta_\ell) \supseteq \mathbb{Q}(\zeta_\ell)$. The Galois group $\text{Gal}(\mathbb{Q}(\zeta_{\ell^{b_\ell}})/\mathbb{Q}(\zeta_\ell))$ is cyclic of order $\ell^{b_\ell-1}$. We deduce then that $A_{\ell^{b_\ell}}\mathbb{Q}(\zeta_\ell) = \mathbb{Q}(\zeta_{\ell^{j_0}})$ for some $1 \leq j_0 \leq b_\ell$. However, since m_K is minimal, b_ℓ must be minimal as well. Therefore, we must have that $A_{\ell^{b_\ell}} \not\subseteq \mathbb{Q}(\zeta_{\ell^{b_\ell-1}})$. This implies that $A_{\ell^{b_\ell}}\mathbb{Q}(\zeta_\ell) = \mathbb{Q}(\zeta_{\ell^{b_\ell}})$. Thus, from the diagram (9), we see that $\varphi_K(\ell) = \varphi_K(\ell^{b_\ell})$. The claim in (8) follows since $\varphi_K(\ell^j)$ divides $\varphi_K(\ell^{j+1})$ for all $j \geq 1$. The lemma follows by combining (7) with (8). \square

The final goal of this section is to study the Dirichlet generating function

$$D_K(s) := \sum_{n=1}^{\infty} \frac{1}{\varphi_K(n)n^{s-1}}$$

and use it to prove two asymptotic identities involving the function $\varphi_K(n)$. Since $\varphi_K(n)$ agrees with $\varphi(n)$ for $\gcd(n, m_K) = 1$, we begin with the Dirichlet series

$$D(s) := \sum_{n=1}^{\infty} \frac{1}{\varphi(n)n^{s-1}}$$

and introduce finitely many correction factors to obtain $D_K(s)$. Let $h(s)$ denote the Euler product

$$h(s) := \prod_{\ell} \left\{ 1 + \frac{1}{\ell^{s+2}} \left(1 - \frac{1}{\ell^s} \right) \left(1 - \frac{1}{\ell} \right)^{-1} \right\};$$

and observe that, for any $\epsilon > 0$, $h(s)$ is holomorphic and bounded for $\operatorname{Re}(s) > -\frac{1}{2} + \epsilon$. Using the product formula for Euler's φ function, we factor $D(s)$ as

$$D(s) = \prod_{\ell} \left\{ 1 + \frac{1}{\ell^s} \left(1 - \frac{1}{\ell}\right)^{-1} \left(1 - \frac{1}{\ell^s}\right)^{-1} \right\} = \zeta(s)\zeta(s+1)h(s), \quad (10)$$

where again $\zeta(s)$ is the Riemann zeta function.

We now return to the Dirichlet series $D_K(s)$. In light of (10) and Lemma 1, for each prime ℓ dividing m_K , we define the correction factor

$$D_{K,\ell}(s) := \frac{\left\{ 1 + \frac{1}{\varphi_K(\ell)\ell^{s-1}} \left(1 - \left(\frac{1}{\ell^{s-1}}\right)^{b_{\ell}-1}\right) \left(1 - \frac{1}{\ell^{s-1}}\right)^{-1} + \frac{1}{\varphi_K(\ell)} \left(\frac{1}{\ell^{s-1}}\right)^{b_{\ell}} \left(1 - \frac{1}{\ell^s}\right)^{-1} \right\}}{\left\{ 1 + \frac{1}{\ell^s} \left(1 - \frac{1}{\ell}\right)^{-1} \left(1 - \frac{1}{\ell^s}\right)^{-1} \right\}},$$

which has removable singularities at $s = 0, 1$ and is analytic elsewhere. We also define $D_{K,\ell}(0)$ (resp. $D_{K,\ell}(1)$) to be the limit of $D_{K,\ell}(s)$ as s approaches 0 (resp. 1). In particular, we note that

$$D_{K,\ell}(0) = \lim_{s \rightarrow 0} D_{K,\ell}(s) = \frac{\varphi(\ell^{b_{\ell}})}{\varphi_K(\ell)} = \frac{\varphi(\ell^{b_{\ell}})}{\varphi_K(\ell^{b_{\ell}})}. \quad (11)$$

Finally, from (10), we observe that $D_K(s)$ may be factored as

$$D_K(s) = \zeta(s)\zeta(s+1)h(s) \prod_{\ell|m_K} D_{K,\ell}(s). \quad (12)$$

Lemma 2. *For a fixed number field K , we have*

$$\sum_{n < x} \left(1 - \frac{n}{x}\right)^2 \frac{1}{\varphi_K(n)} = c_1 \log x + c_2 + \frac{\varphi(m_K)}{\varphi_K(m_K)} \frac{\log x}{x} + \frac{c_3}{x} + O\left(x^{-\frac{5}{4}}\right); \quad (13)$$

$$\sum_{n \leq x} \frac{1}{\varphi_K(n)} = c_1 \log x + c_4 + O\left(\frac{1}{x}\right), \quad (14)$$

where $c_1 = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \prod_{\ell|m_K} D_{K,\ell}(1)$, and c_2, c_3, c_4 are constants.

Proof. We begin with the proof of (13). For $c > 0$,

$$\begin{aligned} \frac{1}{2} \sum_{n < x} \left(1 - \frac{n}{x}\right)^2 \frac{1}{\varphi_K(n)} &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} D_K(s+1) \frac{x^s}{s(s+1)(s+2)} ds \\ &= R_0 + R_{-1} + \frac{1}{2\pi i} \int_{-\frac{5}{4}-i\infty}^{-\frac{5}{4}+i\infty} D_K(s+1) \frac{x^s}{s(s+1)(s+2)} ds, \end{aligned}$$

where R_0 and R_{-1} are the residues of the integrand at $s = 0$ and $s = -1$ respectively. See [11, Exercise 4.1.9, p. 57] for example. Using (12), we calculate the residues as follows:

$$\begin{aligned} R_0 &= \frac{\zeta(2)h(1) \prod_{\ell|m_K} D_{K,\ell}(1)}{2} \log x + \frac{1}{2} c_2 = \frac{c_1}{2} \log x + \frac{1}{2} c_2; \\ R_{-1} &= \frac{-\zeta(0)h(0) \prod_{\ell|m_K} D_{K,\ell}(0) \log x}{x} + \frac{c_3}{2x} = \frac{\varphi(m_K)}{\varphi_K(m_K)} \frac{\log x}{2x} + \frac{c_3}{2x}, \end{aligned}$$

where we have applied (11) to compute $\prod_{\ell|m_K} D_{K,\ell}(0)$. The remaining integral is clearly $O(x^{-5/4})$.

For the proof of (14), we begin with the formula

$$\sum_{n \leq x} \frac{1}{\varphi_K(n)} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} D_K(s+1) \frac{x^s}{s} ds$$

and proceed in a manner similar to the proof of (13). \square

5. PROOF OF THEOREM 2

Let $\theta_K(x) := \sum_{N\mathfrak{p} \leq x} \log N\mathfrak{p}$. We will frequently make use of the formula

$$\theta_K(x) = x + O(x/(\log x)^M) \quad (15)$$

throughout the remainder of the article. The formula follows from (4). We now begin the proof of Theorem 2 by stating and proving the following lemma.

Lemma 3. *For any $M > 0$,*

$$\sum_{N\mathfrak{p} \leq x} \sum_{N\mathfrak{p}' = N\mathfrak{p}} (\log N\mathfrak{p})^2 = [K : \mathbb{Q}] (x \log x - x) + O\left(\frac{x}{(\log x)^M}\right).$$

Proof. First, note that since only finitely many rational primes may ramify in K , we only introduce an error of $O(1)$ by restricting our sum to prime ideals which do not lie above a rational prime ramifying in K . For a rational prime p , let g_p denote the number of primes lying above p , let f_p denote the degree of any prime lying above p , and let e_p denote the ramification index of p in K . Note that e_p and f_p are well-defined since K/\mathbb{Q} is Galois. The contribution from the degree one primes gives us our main term. Thus, partial summation and (15) yield

$$\begin{aligned} \sum_{N\mathfrak{p} \leq x} \sum_{N\mathfrak{p}' = N\mathfrak{p}} (\log N\mathfrak{p})^2 &= [K : \mathbb{Q}] \sum_{\substack{p \leq x \\ e_p = 1 \\ f_p = 1}} g_p (\log p)^2 + O(\sqrt{x} \log x) \\ &= [K : \mathbb{Q}] \log x (\theta_K(x) + O(\sqrt{x})) - [K : \mathbb{Q}] \int_1^x \frac{\theta_K(t) + O(\sqrt{t})}{t} dt \\ &= [K : \mathbb{Q}] (x \log x - x) + O(x(\log x)^{-M}). \end{aligned}$$

\square

Proof of Theorem 2. First, define

$$S(x; Q_1, Q_2) := \sum_{Q_1 < q \leq Q_2} \sum_{a \in G_q} \left(\theta_K(x; q, a) - \frac{x}{\varphi_K(q)} \right)^2.$$

If $Q \leq x(\log x)^{-(M+1)}$, then Theorem 1 implies that $S(x; 0, Q) \ll x^2(\log x)^{-M}$, and hence Theorem 2 follows since the error term dominates in this case. Thus, it suffices to consider the case when $Q > x(\log x)^{-(M+1)}$. Therefore, for the remainder of the proof, we will write $Q_1 := x(\log x)^{-(M+1)}$, and assume that $Q_1 < Q_2 \leq x$. By Theorem 1, we have

$$S(x; 0, Q_2) = S(x; Q_1, Q_2) + O(x^2(\log x)^{-M}). \quad (16)$$

For Q_1, Q_2 as above,

$$\begin{aligned}
S(x; Q_1, Q_2) &= \sum_{Q_1 < q \leq Q_2} \sum_{a \in G_q} \left\{ \theta_K(x; q, a)^2 - \frac{2x}{\varphi_K(q)} \theta_K(x; q, a) + \frac{x^2}{\varphi_K(q)^2} \right\} \\
&= \sum_{Q_1 < q \leq Q_2} \left\{ \sum_{a \in G_q} \theta_K(x; q, a)^2 - \frac{x}{\varphi_K(q)} \left(2\theta_K(x) - 2 \sum_{\substack{N\mathfrak{p} \leq x, \\ (N\mathfrak{p}, q) > 1}} \log N\mathfrak{p} - x \right) \right\} \\
&= \sum_{Q_1 < q \leq Q_2} \sum_{a \in G_q} \theta_K(x; q, a)^2 - x^2 \sum_{Q_1 < q \leq Q_2} \frac{1}{\varphi_K(q)} + O\left(\frac{x^2}{(\log x)^M}\right). \tag{17}
\end{aligned}$$

Now, observe that

$$\begin{aligned}
\sum_{a \in G_q} \theta_K(x; q, a)^2 &= \sum_{\substack{N\mathfrak{p}, N\mathfrak{p}' \leq x, \\ N\mathfrak{p} \equiv N\mathfrak{p}' \pmod{q}, \\ (\mathfrak{p}\mathfrak{p}', q\mathcal{O}_K) = 1}} \log N\mathfrak{p} \log N\mathfrak{p}' \\
&= \sum_{\substack{N\mathfrak{p} = N\mathfrak{p}' \leq x, \\ (\mathfrak{p}\mathfrak{p}', q\mathcal{O}_K) = 1}} (\log N\mathfrak{p})^2 + \sum_{\substack{N\mathfrak{p}, N\mathfrak{p}' \leq x; \\ N\mathfrak{p} \neq N\mathfrak{p}', \\ N\mathfrak{p} \equiv N\mathfrak{p}' \pmod{q}}} \log N\mathfrak{p} \log N\mathfrak{p}'.
\end{aligned}$$

Note that removing the condition $(\mathfrak{p}\mathfrak{p}', q\mathcal{O}_K) = 1$ from the second sum is justified. For example, if $\mathfrak{p}|q\mathcal{O}_K$ and p lies below \mathfrak{p} , then the condition $N\mathfrak{p} \equiv N\mathfrak{p}' \pmod{q}$ implies that $0 \equiv N\mathfrak{p}' \pmod{p}$. This in turn implies that $N\mathfrak{p} = N\mathfrak{p}'$. Thus, we define

$$\begin{aligned}
H(x; Q_1, Q_2) &:= \sum_{Q_1 < q \leq Q_2} \sum_{\substack{N\mathfrak{p} = N\mathfrak{p}' \leq x, \\ (\mathfrak{p}\mathfrak{p}', q\mathcal{O}_K) = 1}} (\log N\mathfrak{p})^2; \\
J(x; Q_1, Q_2) &:= \sum_{Q_1 < q \leq Q_2} \sum_{\substack{N\mathfrak{p}, N\mathfrak{p}' \leq x; \\ N\mathfrak{p} \neq N\mathfrak{p}', \\ N\mathfrak{p} \equiv N\mathfrak{p}' \pmod{q}}} \log N\mathfrak{p} \log N\mathfrak{p}'.
\end{aligned}$$

Now (17) may be rewritten as

$$\begin{aligned}
S(x; Q_1, Q_2) &= H(x; Q_1, Q_2) + J(x; Q_1, Q_2) \\
&\quad - c_1 x^2 \log(Q_2/Q_1) + O\left(\frac{x^2}{(\log x)^M}\right). \tag{18}
\end{aligned}$$

Note that we have applied the second part of Lemma 2 to the second term of (17).

Removing the condition $(\mathfrak{p}\mathfrak{p}', q\mathcal{O}_K) = 1$ from the inner sum of $H(x; Q_1, Q_2)$ introduces an error which is $O((\log x)^2)$. Thus, we may apply Lemma 3 to obtain

$$\begin{aligned}
H(x; Q_1, Q_2) &= \{Q_2 - Q_1 + O(1)\} \{[K : \mathbb{Q}](x \log x - x) + O(x(\log x)^{-M})\} \\
&= [K : \mathbb{Q}]xQ_2 \log x - [K : \mathbb{Q}]xQ_2 + O\left(\frac{x^2}{(\log x)^M}\right). \tag{19}
\end{aligned}$$

Now, define $J(x; Q) := J(x; Q, x)$, so that $J(x; Q_1, Q_2) = J(x; Q_1) - J(x; Q_2)$. Then

$$\begin{aligned} J(x; Q) &= 2 \sum_{\substack{\mathbf{Np}' < \mathbf{Np} \leq x, \\ \mathbf{Np} - \mathbf{Np}' = kQ, \\ Q < q \leq x}} \log \mathbf{Np} \log \mathbf{Np}' = 2 \sum_{k < x/Q} \sum_{\substack{\mathbf{Np} \equiv \mathbf{Np}' \pmod{k}, \\ \mathbf{Np} \leq x; \mathbf{Np} - \mathbf{Np}' > kQ}} \log \mathbf{Np} \log \mathbf{Np}' \\ &= 2 \sum_{k < x/Q} \sum_{a \in G_k} \sum_{\substack{\mathbf{Np}' < x - kQ, \\ \mathbf{Np}' \equiv a \pmod{k}}} \log \mathbf{Np}' \sum_{\substack{kQ + \mathbf{Np}' < \mathbf{Np} \leq x, \\ \mathbf{Np} \equiv a \pmod{k}}} \log \mathbf{Np}. \end{aligned}$$

Since $Q \geq Q_1 = x/(\log x)^{M+1}$, we have $k < x/Q \leq (\log x)^{M+1}$ and $kQ \geq x/(\log x)^{M+1}$. Thus, we may apply (4) and write

$$\theta_K(x; a, k) - \theta_K(kQ + \mathbf{Np}'; a, k) = \frac{x - kQ - \mathbf{Np}'}{\varphi_K(k)} + O\left(\frac{x}{(\log x)^{2M+1}}\right)$$

for the innermost sum above. This gives

$$\begin{aligned} J(x, Q) &= 2 \sum_{k < \frac{x}{Q}} \frac{1}{\varphi_K(k)} \sum_{\substack{\mathbf{Np}' < x - kQ, \\ (\mathbf{Np}', k) = 1}} (x - kQ - \mathbf{Np}') \log \mathbf{Np}' + O\left(\frac{x}{(\log x)^{2M+1}} \sum_{k < \frac{x}{Q}} \theta_K(x)\right) \\ &= 2 \sum_{k < \frac{x}{Q}} \frac{\int_1^{x-kQ} \theta_K(t) dt}{\varphi_K(k)} + O\left(x \sum_{k < \frac{x}{Q}} \frac{\log k}{\varphi_K(k)}\right) + O\left(\frac{x^3}{Q(\log x)^{2M+1}}\right), \end{aligned}$$

where the last line follows by partial summation applied to the inner sum of the main term. Therefore, by (15), we have

$$J(x, Q) = x^2 \sum_{k < \frac{x}{Q}} \left(1 - \frac{kQ}{x}\right)^2 \frac{1}{\varphi_K(k)} + O\left(\frac{x^2}{(\log x)^M}\right).$$

We consider two different cases for the treatment of $J(x; Q_1, Q_2)$. First, if $Q_2 = x$, then

$$\begin{aligned} J(x; Q_1, Q_2) &= J(x; Q_1) \\ &= x^2 \left\{ c_1 \log(x/Q_1) + c_2 + O\left(\frac{\log(x/Q_1)}{x/Q_1}\right) \right\} + O\left(\frac{x^2}{(\log x)^M}\right) \\ &= c_1 x^2 \log(Q_2/Q_1) + c_2 x^2 + O\left(\frac{x^2}{(\log x)^M}\right). \end{aligned} \tag{20}$$

In the case that $Q_2 \leq x$ (including the previous case), we may write

$$\begin{aligned} J(x; Q_1, Q_2) &= J(x; Q_1) - J(x; Q_2) \\ &= c_1 x^2 \log(Q_2/Q_1) - \frac{\varphi(m_K)}{\varphi_K(m_K)} x Q_2 \log(x/Q_2) - c_3 x Q_2 \\ &\quad + O\left(x^{3/4} Q_2^{5/4}\right) + O\left(\frac{x^2}{(\log x)^M}\right). \end{aligned} \tag{21}$$

Theorem 2 now follows by combining (16), (18), (19), (20), and (21). \square

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