Test Help
A tour of some ideas

Warning: not certified as typo-free. If it looks like it must be a typo, it very well might be a typo. All typos are the responsibility of the Spook from Scary Movie. In several days, an updated and vastly extended version of this worksheet will be available at www.math.mtu.edu/~daolson.

1. Is there a copy of $\mathbb{Z}_2$, somewhat disguised, hiding inside $\mathbb{Z}_{10}$?

   (a) Fill out the group table, using $a = 0$ and $b = 1$ in $\mathbb{Z}_2$.

   $$
   \begin{array}{c|cc}
   + & a & b \\
   \hline
   a & & \\
   b & & 
   \end{array}
   $$

   (b) Fill out the group table, using $a = 0$ and $b = 5$ in $\mathbb{Z}_{10}$.

   $$
   \begin{array}{c|cc}
   + & a & b \\
   \hline
   a & & \\
   b & & 
   \end{array}
   $$

   (c) Make the (hopefully) obvious conclusion. If we want to use the map $\phi : \mathbb{Z}_2 \to \mathbb{Z}_{10}$ to show how $\mathbb{Z}_2$ is undercover in $\mathbb{Z}_{10}$, we would let $\phi(0) = 0$. What should $\phi(1)$ be?

   (d) Because we want $\phi$ to preserve group actions, we must have $\phi(0) = \phi(0 + 0) = \phi(0) + \phi(0)$. In your first high school algebra class, you probably learned how to solve $x = x + x$ for $x$. Is this situation any different?

2. Repeat: Is there a copy of $\mathbb{Z}_3$, somewhat disguised, hiding inside $\mathbb{Z}_{12}$?

   (a) Fill out the group table, using $a = 0$ and $b = 1$ in $\mathbb{Z}_3$ ($c$ must be . . .).

   $$
   \begin{array}{c|ccc}
   + & a & b & c \\
   \hline
   a & & & \\
   b & & & \\
   c & & & 
   \end{array}
   $$

   (b) Fill out the group table, using $a = 0$ and $b = 4$ in $\mathbb{Z}_{12}$ ($c$ must be . . .).

   $$
   \begin{array}{c|ccc}
   + & a & b & c \\
   \hline
   a & & & \\
   b & & & \\
   c & & & 
   \end{array}
   $$

   (c) Make the (again, hopefully) obvious conclusion. If we want to use the map $\phi : \mathbb{Z}_3 \to \mathbb{Z}_{12}$ to show how $\mathbb{Z}_3$ is undercover in $\mathbb{Z}_{12}$, we would let $\phi(0) = 0$. What should $\phi(1)$ be?
(d) Because we want the map \( \phi \) to preserve group actions, we know that \( \phi(2) = \phi(1 + 1) = \phi(1) + \phi(1) \). Does this work?

(e) What if some twisted soul wanted to use \( \varphi(1) = 8 \). What must \( \varphi(2) \) be, if \( \varphi \) preserves the group action?

(f) Try to fill out the group table, using \( a = 0 \) and \( b = 8 \) in \( \mathbb{Z}_{12} \).

<table>
<thead>
<tr>
<th>+</th>
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3. Take another look at Problem 1 from the test. Saying that \( \phi \) is an injective homomorphism is just saying that \( \mathbb{Z}_2 \) is hiding out, in disguise, in \( \mathbb{Z}_{1000} \). Phrased that way, is the answer now obvious?

4. Saying that \( \phi : \mathbb{Z}_2 \rightarrow \mathbb{Z}_{1000} \) is a homomorphism just means that it preserves group actions: \( \phi(a + b) = \phi(a) + \phi(b) \). What does it mean for it to be injective?

5. Find a homomorphism \( \phi : \mathbb{Z}_2 \rightarrow \mathbb{Z}_{1000} \) which is not injective. Hint: you don’t have many choices.

6. Is there a function \( \phi : \mathbb{Z}_2 \rightarrow \mathbb{Z}_{1000} \) which is surjective? Give an example or (preferably) explain why it is impossible.

7. Suppose \( \phi : \mathbb{Z}_5 \rightarrow \mathbb{Z}_{10} \) is a homomorphism.

   (a) What must \( \phi(0) \) be? Can you recreate the argument given earlier in this worksheet, without peeking?

   (b) Assume that \( \phi(1) = 4 \). Calculate \( \phi(2) = \phi(1 + 1) = \phi(1) + \phi(1) \), and use a similar process to calculate \( \phi(3) \) and \( \phi(4) \).

   (c) Check whether \( \phi(1 + 3) = \phi(1) + \phi(3) \).

   (d) Check whether \( \phi(2 + 3) = \phi(2) + \phi(3) \).

   (e) Check whether \( \phi(4 + 3) = \phi(4) + \phi(3) \).

   (f) Obviously 1 generates \( \mathbb{Z}_5 \). Does 4 generate \( \mathbb{Z}_{10} \)? Does 4 generate a disguised copy of \( \mathbb{Z}_5 \) hiding in \( \mathbb{Z}_{10} \)?

   (g) Does 6 generate the disguised copy of \( \mathbb{Z}_5 \) hiding in \( \mathbb{Z}_{10} \)? What does that answer tell you about what would have happened if \( \phi(1) = 6 \)?

8. Gilligan says that a homomorphism \( \phi : \mathbb{Z}_{15} \rightarrow \mathbb{Z}_5 \), where \( \phi(1) = 1 \), is like a triple–layer chocolate cake. Would you agree? Is it surjective? Is it injective?

9. Suppose that \( H \) is a normal subgroup of a group \( G \). Let \( h_1 \in H \) and \( h_2 \in H \). What can you say about the product \( h_1 h_2 \) if \( h_1 \neq h_2 \)? Would it make any difference if they were the same element?

10. Suppose that \( H \) is a normal subgroup of a group \( G \). Suppose \( h \in H \). Could you say the same thing about \( h^2 \)? How about \( h^3 \)? And \( h^{67} \)? Could you prove that \( h^n \in H \) for any integer \( n \)? (Negative integers too!)
11. Let \( g \in G \), a group, and \( h \in H \), a normal subgroup. What can you say about \( ghg^{-1} \)? What about \( g^{-1}hg \)? As a misdirection question, can you say anything about \( hgh^{-1} \)?

12. Let \( g \in G \), a abelian group. Suppose \( h \in H \), a subgroup. Is the subgroup \( H \) abelian? Is it normal?

13. Let \( g \in G \), a group, and \( h_1 \in H \), a normal subgroup. Lunatic Larry says that elements in a normal subgroup “almost” commute, because \( gh_1 = h_2g \), where \( h_2 \in H \). Lunacy? Or freakishly correct?

14. Add \( 1/5 \) and \( 1/3 \). Would it make any difference if you added \( 3/15 \) and \( 5/15 \) instead?

15. Let \( k \) and \( l \) be integers. If you added \( k \cdot 1/5 + l \cdot 1/3 \), Could you always express the result as \( m/15 \), where \( m \) is some integer?

16. Can you put \( 3/23, 34/42, 38/n \) over a common denominator? No matter how many times you added or subtracted those numbers, would you ever need a larger common denominator?